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Finite BN-Pairs of Rank Two and Even Characteristic, Having a Non-Trivial Cartan Subgroup

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1. INTRODUCTION

Let G be a simple Chevalley group, Steinberg variation, or Ree group, defined over a finite field $GF(2^n)$, $n > 1$, and having Lie rank two. The condition $n > 1$ guarantees the existence of a non-identity Cartan subgroup H of G, and H occupies an important position with respect to both the 2-local and the p-local structure of G, for certain odd primes p.

Let $\mathscr{T}$ be the set of maximal (under inclusion) H-invariant 2-subgroups of G. (This definition, and others following, are provisional, and are intended to motivate the definitions given in a more abstract setting a little further on). Then we have $G = \langle \mathscr{T} \rangle$. Let $\mathscr{P}^*$ denote the set of proper subgroups of G, other than 2-groups, which are generated by members of $\mathscr{T}$. Then $\{N(P): P \in \mathscr{P}^*\}$ is the set of maximal parabolic subgroups of G which contain H, and $P/O_2(P)$ is a quasisimple Bender broup for any $P \in \mathscr{P}^*$. Moreover, every maximal 2-local subgroup M of G is of the form $N(P^g)$ for some $g \in G$ and some $P \in \mathscr{P}^*$. This indicates the connection between H and the 2-local structure of G.

At the same time, if p is a prime dividing $2^n - 1$ then $H = C_G(D)$, where $D = \Omega_1(O_p(H))$ is an elementary abelian p-group of p-rank two (the single exception occuring when $G \cong L_3(4)$ and $p = 3$). Thus $N_G(H) = N_G(D)$ is a p-local subgroup of G. The group $W = N_G(H)/H$ is the Weyl group of G, and we have $G = \langle T, N_G(H) \rangle$ for any $T \in \mathscr{T}$. This connection between 2-local and p-local structure is the basis for the global structure of G via its BN-pair.

We shall formulate a set of axioms for the situation described above. Thus, let G now denote an arbitrary (to begin with) finite group, having an elementary abelian p-subgroup D of p-rank two. Set $H = C_G(D)$, and assume that N normalizes some non-identity 2-subgroup of G.

Let $\mathscr{T}^* = \mathrm{И}^*(H; 2)$ be the set of maximal H-invariant 2-subgroups of G under inclusion. A subset $\mathscr{T}$ of $\mathscr{T}^*$ will be said to be *complete* if whenever $S \in \mathscr{T}$ and $T \in \mathscr{T}^*$ with $S \cap T \neq 1$, then $T \in \mathscr{T}$. We say that $\mathscr{T}$ is a *component*

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of $\mathscr{T}^*$ if $\mathscr{T}$ is complete and the only proper complete subset of $\mathscr{T}$ is the empty set.

Fix a component $\mathscr{T}$ of $\mathscr{T}^*$. For any subgroup X of G set $\mathscr{T}(X) = \{R \in И_X^*(H; 2): R \leqslant T \text{ for some } T \in \mathscr{T}\}$. Let $\mathscr{Y}$ be the collection of all subgroups Y of G such that

(i) $1 < O_2(Y) < Y$, and
(ii) $Y = \langle \mathscr{T}(Y) \rangle$.

Let $\mathscr{P}$ be the collection of all P in $\mathscr{Y}$ such that $P/O_2(P)$ is a quasisimple Bender group, and let $\mathscr{P}^*$ be the set of maximal members of $\mathscr{P}$, under inclusion. We consider the following

Hypothesis

$$H1: \mathscr{Y} = \mathscr{P}.$$
$$H2: \mathscr{T}(P) \leqslant \mathscr{T} \qquad \text{for all } P \in \mathscr{P}^*.$$
$$H3: N_P(H) \leqslant N(D) \qquad \text{for all } P \in \mathscr{P}.$$
$$H4: O_2(\langle \mathscr{T} \rangle) = 1.$$

We shall establish the following two theorems.

THEOREM A. *Assume the Hypothesis. Let $T \in \mathscr{T}$, set $B = (H \cap \langle \mathscr{T} \rangle)\, T$, and set $N = \langle N_p(H): P \in \mathscr{P} \rangle$. Then either*

(i) *$C_D(\langle \mathscr{T} \rangle) \neq 1$ and T is isomorphic to a Sylow 2-subgroup of $L_3(2^n)$ for some fixed n, $n > 1$, or*

(ii) *$C_D(\langle \mathscr{T} \rangle) = 1$ and $\langle \mathscr{T} \rangle = \langle B, N \rangle$ is a split BN-pair of rank two.*

THEOREM B. *In addition to the Hypothesis, assume that G is of characteristic 2-type, and that $N_G(P)$ is a maximal 2-local subgroup of G for all $P \in \mathscr{P}^*$. Then $\langle \mathscr{T} \rangle = F^*(G)$ is isomorphic to $L_3(2^n)$, $S_p(4, 2^n)$, $G_2(2^n)$, ${}^2F_4(2^n)$, ${}^3D_4(2^n)$, $U_4(2^n)$, or $U_5(2^n)$ for some n, $n > 1$.*

Theorems A and B are intended as a step towards the classification of quasithin simple groups (simple groups G such that $m_p(M) \leqslant 2$ for all odd primes p and all 2-local subgroups M of G). With finitely many exceptions, the known quasithin simple groups of characteristic 2-type are among the groups in the conclusion of Theorem B. In some sense these theorems are inspired by and are analogous to the results in [1], which have their main application to the classification [2] of thin simple groups. Most of the ideas used in proving theorem A come from papers of Sims [12] and Glauberman [6] by way of David Goldschmidt. Indeed, we would like to express our gratitude to Professor Goldschmidt for sharing his ideas on extending the Glauberman–Sims construction (which appear here in section 4). This new construction seems also to have been anticipated by Richard Niles, in [10].

Notation. Set $N = \langle N_P(H): P \in \mathscr{P} \rangle$, and put $W = NH/H$. For X an H-invariant subset or element of G and for $w \in W$ we define $X^w = X^n$, for any element n of the coset w. For X an H-invariant subgroup of G we write $X \cap W$ or $W \cap X$ for $(X \cap N) H/H$.

Define a (symmetric) relation Δ on $\mathscr{P}$ by $(P_0, P_1) \in \Delta$ if $|\mathscr{T}(P_0) \cap \mathscr{T}(P_1)| = 1$. If in addition we have $\{P_0, P_1\} \leqslant \mathscr{P}^*$, we write $(P_0, P_1) \in \Delta^*$.

For $T \in \mathscr{T}$, set $\mathscr{P}(T) = \{P \in \mathscr{P}: T \leqslant P\}$.

For $d \in D^\#$, set $\mathscr{T}_d = \mathscr{T}(C_G(d))$, $W_d = C_G(d) \cap W$, $W_{\langle d \rangle} = N_G(\langle d \rangle) \cap W$, $\mathscr{P}_d = \{P \in \mathscr{P}: P \leqslant C(d)\}$, and for $R \in \mathscr{T}_d$, set $\mathscr{P}_d(R) = \{P \in \mathscr{P}_d : R \leqslant P\}$.

An H-invariant subgroup X of G is said to be of *type* $L_3(2^n)$ if $X = \langle \mathscr{T}(X) \rangle \neq 1$, $O_2(X) = 1$, and every member of $\mathscr{T}(X)$ is of type $L_3(2^n)$.

If V is an elementary abelian 2-group and X is a group of automorphisms of V with Sylow 2-subgroup S, such that $X \cong L_2(2^n)$, $|V| = 2^{2n}$, and $|C_V(S)| = 2^n$, we say that V is a *natural module* for X.

For S a 2-group, $\mathscr{A}(S)$ denotes the collection of elementary abelian subgroups of S of maximal rank, $J(S) = \langle \mathscr{A}(S) \rangle$, $\tilde{Z}(S) = \Omega_1(Z(J(S)))$, and $\tilde{J}(S) = C_S(\tilde{Z}(S))$.

2. Bender Groups on 2-Groups

In this preliminary section, we depart briefly from the notation given in the introduction, and consider groups G such that $G/O_2(G)$ is isomorphic to $L_2(2^n)$, $Sz(2^n)$, $SU(3, 2^n)$, or $U_3(2^n)$, for some n, $n \geqslant 1$. We fix $q = 2^n$. The results of this section will only be needed in the case that $n > 1$, but since no extra argument is required in the case $n = 1$, we include it.

(2.1) Lemma. *Suppose $\tilde{J}(S) \nleqslant O_2(G)$ for $S \in \mathrm{Syl}_2(G)$, $\Omega_1(Z(S)) \leqslant O_2(G)$, and $\Omega_1(Z(S)) \ntrianglelefteq G$. Then $G/O_2(G) \cong L_2(q)$ and, setting $V = \Omega_1(Z(O_2(G)))$, $V/C_V(G)$ is natural module for $G/O_2(G)$.*

Proof. Set $Z = \Omega_1(Z(S))$. Then $Z < V$, as $Z \ntrianglelefteq G$. Since $\tilde{J}(S) \ntrianglelefteq O_2(G)$, $V \nleqslant \tilde{Z}(S)$, and so $J(S) \nleqslant O_2(G)$. Let $A \in \mathscr{A}(S)$, with $A \nleqslant O_2(G)$. By the maximality of A, we have $|A| \geqslant |(A \cap O_2(G))V|$, so that $|A|\,|A \cap V| \geqslant |A \cap O_2(G)|\,|V|$. Setting $\bar{G} = G/O_2(G)$, we obtain $|\bar{A}| = |A: A \cap O_2(G)| \geqslant |V: A \cap V| = |V/C_V(A)|$.

It is well-known that three conjugates of $\bar{A}$ suffice to generate $\bar{G}$, and that two conjugates suffice if $|\bar{A}| > 2$ and $\bar{G}$ is not a unitary group. Now $|\bar{A}| \leqslant q$, so that $|V: C_V(G)| \leqslant q^3$. But $|Sz(q)| \nmid |L_m(2)|$ for $m < 4n$ and $|U_3(q)| \nmid |L_m(2)|$ for $m < 6n$, and so $\bar{G} \cong L_2(q)$. Suppose $|\bar{A}| = 2$. Then $|V: C_V(G)| \leqslant 8$, so that $q = 2$. Thus we are now reduced to the case in which $\bar{G}$ is generated by two conjugates of $\bar{A}$. Hence $|V: C_V(G)| \leqslant q^2$, and since $|L_2(q)| \nmid L_m(2)|$ for $m > 2n$, we have $|V: C_V(G)| = q^2$ and $|\bar{A}| = q$. Thus $S = O_2(G)A$ and

$|V: C_V(A)| = |V: C_V(S)| = q$. Thus $V/C_V(G)$ is a natural module for $\bar{G}$, which proves (2.1).

We record also:

(2.2) LEMMA. *Let V be a nontrivial $Z_2[G]$-module, with $|V| \leqslant q^2$. Then $G/O_2(G) \cong L_2(q)$, and $|V| = q^2$.*

(2.3) LEMMA *Let V be an irreducible $Z_2[G]$-module, $G \cong L_2(q)$, such that $[V, S] \neq [V, S, S] = 1$ for $S \in \mathrm{Syl}_2(G)$. Then V is a natural module for G.*

Proof. Set $Z = [V, S]$, Let H be a complement in $N_G(S)$ to S, and let g be an involution in $N_G(H)$, with $g \notin S$ if $H = 1$. Then ZZ^g admits $\langle S, g\rangle = G$, so that $V = ZZ^g$, and $Z \cap Z^g \subseteq C_V(\langle S, S^g\rangle) = C_V(G) = 1$. Also $C_V(S) \cap C_V(S^g) = 1$, so $Z = C_V(S)$, as $m(Z) = m(V)/2$.

Let $x \in S^{\#}$. Then $G = \langle S, x^g\rangle$ centralizes $C_V(S) \cap C_V(x^g)$, so $C_V(x) = Z$. Let x' be an involution in $G - S$. Then $C_V(x') \cap C_V(x)$ centralizes $\langle S, x'\rangle = G$, so that $C_V(\langle x, x'\rangle) = 1$, and so $C_V(xx') = 1$. This shows that every element of odd order in G is fixed-point-free on V.

Choose $x \in S^{\#}$ so that $|xg| = 3$, and set $y = xg$. Let U be a minimal non-identity H-invariant subgroup of Z, and set $W = UU^y$. As $C_V(y) = 1$, it follows that W admits y, and then as W admits $\langle H, g\rangle$ we have $W = V$. Since H is faithful and irreducible on U, we have $|U| = q$. Thus $|V| = q^2$ and V is a natural module for G.

(2.4) THEOREM (Baumann). *Let $G/O_2(G) \cong L_2(q)$, with $F^*(G) = O_2(G)$. Let $S \in \mathrm{Syl}_2(G)$, and assume that no non-identity characteristic subgroup of S is normal in G. Then G has precisely one non-central 2-chief factor, and the nilpotence class of S is two.*

Proof. See [3].

(2.5) LEMMA. *Let $G/O_2(G) \cong L_2(q)$,* with $F^*(G) = O_2(G)$. *Let $S \in \mathrm{Syl}_2(G)$, and assume that $\Omega_1(Z(S)) \ntrianglelefteq G$. Then $\tilde{J}(S) \in \mathrm{Syl}_2(\langle \tilde{J}(S)^G\rangle)$, and if $\tilde{J}(S) \ntrianglelefteq G$ then G centralizes $O_2(G)/(O_2(G) \cap \tilde{J}(S))$.*

Proof. Set $M = O_2(G)$, $V = \Omega_1(Z(M))$, $W = \tilde{Z}(M)$, and $Z = \Omega_1(Z(S))$. Thus $Z \leqslant V \leqslant W$. Since $Z \ntrianglelefteq G$ we have $Z < V$. We may assume that $\tilde{J}(S) \nleqslant G$, whence $V \ntrianglelefteq \tilde{Z}(S)$.

Thus $J(S) \ntrianglelefteq M$, and so there exists $A \in \mathcal{O}\!\ell(S)$ with $A \ntrianglelefteq M$. By (2.1), it follows that $V/C_V(G)$ is a natural module for G/M and that $S = MA$. Then, also $|V: A \cap V| = q$, so that $\mathcal{O}\!\ell(M) \leqslant \mathcal{O}\!\ell(S)$. This yields $J(M) \leqslant J(S)$ whence $\tilde{Z}(S) \leqslant W$ and $\tilde{J}(M) \leqslant \tilde{J}(S)$.

Let $g \in G - N_G(S)$, and set $K = \langle A, A^g\rangle$. Then $G = KM$. Since $(A \cap M) V \in \mathcal{O}\!\ell(M)$, we have $W \leqslant (A \cap M) V$, whence $|W: A \cap W| = q$.

Then $|W: C_W(K)| \leqslant q^2 = |V: C_V(K)|$, so that $W = C_W(K)V$. Hence $\tilde{Z}(S)^x \leqslant \tilde{Z}(S)V$ for all $x \in K$. Setting $R = \tilde{J}(S) \cap M$, we have $R = C_M(\tilde{Z}(S)V)$, whence $K \leqslant N(R)$, and so $R \trianglelefteq G$. Set $\bar{G} = G/R$. Then $\overline{\tilde{J}(S)} \cap \bar{M} = 1$, so that $\bar{A} = \overline{\tilde{J}(S)}$, and $\bar{A} \cap \bar{M} = 1$. Then $[\bar{A}, \bar{M}] = 1$, whence also $[\bar{K}, \bar{M}] = 1$, and as $\bar{A}$ is abelian, we have $\bar{G} = \bar{K} \times \bar{M}$. Thus $\tilde{J}(S) \in \mathrm{Syl}_2(KR)$, where $KR = \langle \tilde{J}(S)^G \rangle$.

(2.6) COROLLARY. *Let $G/O_2(G) \simeq L_2(q)$, $S \in \mathrm{Syl}_2(G)$, with $F^*(G) = O_2(G)$. Assume that $\Omega_1(Z(S)) \not\leqslant G$ and that no non-identity characteristic subgroup of $\tilde{J}(S)$ is normal in G. Then G has just one non-central 2-chief factor, and the nilpotence class of $\tilde{J}(S)$ is two.*

Proof. Immediate from (2.4) and (2.5).

(2.7) LEMMA. *Let G_1 and G_2 be groups satisfying the hypothesis of this section, contained as subgroups of some group. Assume that $\mathrm{Syl}_2(G_1) \cap \mathrm{Syl}_2(G_2)$ has a unique member S, and that $Z(S) \leqslant O_2(G_1) \cap O_2(G_2)$. Assume also that $Z(G_1) = Z(G_2) = 1$, and that $O_2(\langle G_1, G_2 \rangle) = 1$. Then $G_i/0_2(G_i) \simeq L_2(2^n)$ for some fixed n, $n \geqslant 1$, and $\Omega_1(Z(O_2(G_i)))$ is a natural module for $G_i/O_2(G_i)$.*

Proof. Set $V_i = \Omega_1(Z(O_2(G_i)))$. Suppose first that some non-identity characteristic subgroup of $\tilde{J}(S)$ is normal in G_1. Then since $O_2(\langle G_1, G_2 \rangle) = 1$, we have $\tilde{J}(S) \ntrianglelefteq G_2$. Then $J(S) \ntrianglelefteq G_2$, and so as $Z(G_2) = 1$, it follows from (2.1) that $G_2/O_2(G_2) \simeq L_2(2^{n_2})$ for some n_2, and that V_2 is natural module for $G_2/O_2(G_2)$. If also some non-identity characteristic subgroup of $\tilde{J}(S)$ is normal in G_2, then $G_1/O_2(G_1) \simeq L_2(2^{n_1})$ and V_1 is a natural module for $G_1/O_2(G_1)$. But then $|\Omega_1(Z(S))| = 2^{n_1} = 2^{n_2}$, so that we are done in this case.

We may therefore assume, by symmetry, that no non-identity characteristic subgroup of $\tilde{J}(S)$ is normal in G_1. Then $G_1/O_2(G_1) \simeq L_2(2^{n_1})$, G has just one non-central 2-chief factor, and $cl(\tilde{J}(S)) = 2$, by (2.6). Then V_1 is a natural module for $G_1/O_2(G_1)$, by (2.3). Since $Z(G_1) = 1$, $\Phi(O_2(G_1)) = 1$, so that $V_1 = O_2(G_1)$, and $S = J(S)$. Now $cl(S) = 2$, whence $S/O_2(G_2)$ is abelian, so that $G_2/O_2(G_2) \simeq L_2(2^{n_2})$. Also, since $J(S) \not\leqslant O_2(G_2)$, V_2 is a natural module for $G_2/O_2(G_2)$, by (2.1). As above, this yields $n_1 = n_2$, so the lemma is proved.

(2.8) LEMMA. *Let $G \simeq L_2(q)$, V a natural module for G. Then V admits the structure of a $GF(q)[G]$-module.*

Proof. Set $F = GF(q)$, and put $V^* = V \otimes_{Z_2} F$. Then V^* is an $F[G]$-module of dimension $2n$. Let W be an irreducilbe $F[G]$-submodule of V^*. (Such a W exists as $C_{V^*}(G) \simeq C_V(G) = (0)$. Let Γ be the Galois group of F/Z_2. Then Γ operates on V^* in the obvious way, and as a corollary to the Dedekind inde-independance theorem one has $V^* = \sum_{\alpha \in \Gamma} W^\alpha$. By a theorem of Steinberg in [14], we have $W \simeq U^{\alpha_1} \otimes \cdots \otimes U^{\alpha_k}$ for some natural $F[G]$-module U

and some subset $\{\alpha_1, ..., \alpha_k\}$ of Γ, with $\alpha_1, ..., \alpha_k$ distinct. Then $\dim_F(W) = 2^k$, and so $\dim V^* \geqslant (n/k)\, 2^k$. It follows that $k = 1$ or 2.

Suppose first that $k = 1$, so that V^* is a sum of natural $F[G]$-modules. The G-module homomorphism p: $V^* \to V$ given by $p(a \otimes v) = v$ induces then a G-module isomorphism between U and V, yielding (2.8) in this case. So assume $k = 2$. Then $W \cong U \otimes U^\alpha$ for some involution $\alpha \in \Gamma$, and direct computation yields $\dim_F(C_W(S)) = 1$. But then $\dim(C_{V^*}(S)) = n/2$, whereas $\dim(C_{V^*}(S) = m(C_V(S))) = n$; a contradiction. Thus $k \neq 2$, and (2.8) is proved.

(2.9) Lemma. *Let $G/O_2(G) \cong L_2(q)$, $O_2(G)$ a natural module for $G/O_2(G)$, and assume $J(S) \nleqslant O_2(G)$ for $S \in \mathrm{Syl}_2(G)$. Then S is of type $L_3(q)$.*

Proof. Set $V = O_2(G)$, and identify V with a natural $F[G]$-module, $F = GF(2^n)$. Let $A \in \mathcal{O}\!\mathcal{l}(S)$, $A \neq V$. Then $S = AV$, by the proof of (2.1) and then G splits over V, by Gaschutz's theorem. Since all natural $F(G)$-modules are algebraically conjugate, all are G-isomorphic, and so G is isomorphic to the commutator subgroup of a maximal parabolic subgroup of $L_3(2^n)$.

3. Preliminary Results

From now on, assume the main Hypothesis.

(3.1) Lemma. *Let $P \in \mathscr{P}$, $Q = O_2(P)$. Then P/Q is isomorphic to $L_2(2^n)$, $Sz(2^n)$, $SU(3, 2^n)$, or $U_3(2^n)$ for some $n > 1$, and*

(i) *$D \cap P$ is cyclic of order p, where $p \mid 2^n - 1$,*

(ii) *Either $C_D(P/Q) \neq 1$ and $\mid W \cap P \mid = 2$, or else every element of $D - P$ induces a field automorphism on P/Q, in which case $W \cap P$ is dihedral of order $2p$, and*

(iii) *$\mathscr{T}(P) \leqslant \mathrm{Syl}_2(P)$, $|\mathscr{T}(P)| = 2$, $W \cap P$ acts transitively on $\mathscr{T}(P)$ by conjugation, and $Q = \bigcap_{T \in \mathscr{T}(P)} T$.*

Proof. We are given that P/Q is isomorphic to one of the listed groups. Let $T \in \mathscr{T}(P)$. Then $Q < T$ since $P = \langle \mathscr{T}(P) \rangle$. Since Sylow 2-subgroups of P/Q are T.I. sets, we then have $T \in \mathrm{Syl}_2(P)$. Since T admits H, $C_P(D)$ is 2-closed, and it follows that some d in $D\#$ induces a non-identity inner automorphism on P/Q. Hence $p \mid \mid N_P(T) \cap H \mid$.

Let $D \leqslant R \in \mathrm{Syl}_p(PD)$, and set $Z = Z(R) \cap P$. Since $N_P(T)/T$ is cyclic and $[d, P] \nleqslant Q$, we have $[Z, P] \nleqslant Q$. In particular, if $P/Q \cong SU(3, 2^n)$ for some odd n, then $p \neq 3$. Suppose $R \cap P$ is non-cyclic. Then P/Q is a unitary group, $p \mid 2^n + 1$, and $E(C_{P/Q}(ZQ/Q) \cong L_2(2^n)$. But then $C_{R \cap P}(D)$ is non-cyclic, whereas $C_P(D) \leqslant N(T)$; a contradiction.

We conclude that $R \cap P$ is cyclic, whence $p \mid 2^n - 1$. Let $d_1 \in D - \langle d \rangle$. By a theorem of Lang, either D induces inner automorphisms on L/Q and we may pick d_1 so that $[P, d_1] \subseteq Q$, or d_1 induces a field automorphism of order p on P/Q. In either case, set $P_0 = C_P(d_1)$. Then $Z \leqslant P_0$ and there exists $x \in N_{P_0}(Z)$ with $x^2 \in Q$. Then $[x, H] \leqslant H \cap P_0$, so that $x \in N$, whence $x \in N(D)$, by Hypothesis. It follows that $Z \leqslant D$, so that we may take $\langle d \rangle = \Omega_1(Z)$, which proves (i). Moreover, if $C_D(P/Q) \neq 1$, then $\langle x \rangle$ covers $W \cap P$.

Suppose $C_D(P/Q) = 1$. Then $p \mid n$ and $p \mid 2^{n/p} - 1$, whence also $p \mid (2^n - 1)/(2^{n/p} - 1)$. It follows that there exists $y \in N_P(D) - H$ with $y^p \in H$, and then that $W \cap P$ is dihedral of order $2p$. This yields (ii). Part (iii) follows easily from the fact that P is a (B, N) pair of rank one.

(3.2) LEMMA. *Let $S, T \in \mathscr{T}$. Then there exists $\{S_1, \ldots, S_n\} \leqslant \mathscr{T}$ with $S = S_1$, $T = S_n$, and with $\langle S_i, S_{i+1} \rangle \in \mathscr{P}$ for all i, $1 \leqslant i < n$.*

Proof. We write $S \sim T$ if $S = T$ or if the conclusion of (3.2) holds for $\{S, T\}$. Then $\sim$ is an equivalence relation on $\mathscr{T}$. Suppose we have $S \not\sim T$. Among all such pairs, take $\{S, T\}$ so that $S \cap T$ is as large as possible. As $\mathscr{T}$ is a component of $\mathit{И}_G^*(H; 2)$, $S \cap T \neq 1$. Set $Q = S \cap T$, $S_0 = N_S(Q)$, $T_0 = N_T(Q)$, and set $P = \langle S_0, T_0 \rangle$. By the maximality of Q, P is not a 2-group, so $P \in \mathscr{P}$. Choose $P^* \in \mathscr{P}^*$ with $P \leqslant P^*$. Let $S_0 \leqslant S^* \in \mathscr{T}(P^*)$, and let $T_0 \leqslant T^* \in \mathscr{T}(P^*)$. Then $\{S^*, T^*\} = \mathscr{T}(P^*) \leqslant \mathscr{T}$, by (3.1) (iii) and the Hypothesis. Thus $S^* \sim T^*$. But $S \cap S^* \geqslant S_0 > Q$ and $T \cap T^* \geqslant T_0 > Q$, so that $S \sim S^*$ and $T \sim T^*$, by the maximality of Q, and hence $S \sim T$; a contradiction. Hence (3.2) holds if $S \neq T$.

Since $O_2(\langle \mathscr{T} \rangle) = 1$, by Hypothesis, we have $|\mathscr{T}| > 1$. Since $\sim$ is symmetric and transitive, it then follows that (3.2) holds also if $S = T$.

(3.3) COROLLARY. *$|\mathscr{P}(T)| \geqslant 2$ for all $T \in \mathscr{T}$.*

Proof. Immediate from (3.2) and the fact that $O_2(\langle \mathscr{T} \rangle) = 1$.

(3.4) COROLLARY. *W operates transitively on $\mathscr{T}$, by conjugation.*

Proof. Apply (3.2) and (3.1) (iii).

(3.5) LEMMA. *For any $P \in \mathscr{P}$, there exists a unique member of $\mathscr{P}^*$ containing P.*

Proof. Suppose false. Let $P \leqslant P_1 \cap P_2$, where P_1 and P_2 are distinct elements of $\mathscr{P}^*$. Set $Q = O_2(P_1) \cap O_2(P_2)$, and assume P_1 and P_2 to be chosen so that Q is as large as possible. Then $Q \geqslant O_2(P)$. Set $P_3 = \langle \mathscr{T}(N(Q)) \rangle$, and let $P_3 \leqslant P^* \leqslant \mathscr{P}^*$. Then $P \leqslant P^*$. Set $R = O_2(P^*)$. Then $R \geqslant N(Q) \cap O_2(P_i)$ for $i = 1$ and 2. By the maximality of Q we then have $Q = O_2(P_1) = O_2(P_2)$. But then $\langle P_1, P_2 \rangle \in \mathscr{P}$ so that $P_1 = P_2$, as $\{P_1, P_2\} \leqslant \mathscr{P}^*$; a contradiction.

(3.6) LEMMA. $\bigcap_{T\in\mathscr{T}} T = 1$.

Proof. Set $Q = \bigcap_{T\in\mathscr{T}} T$ and assume $Q \neq 1$. Set $P = \langle \mathscr{T}(N(Q))\rangle$. Then P admits W, and it follows from (3.4) that P is not a 2-group, so that $P \in \mathscr{P}$, and $Q = O_2(P)$. By (3.5), we have $P \in \mathscr{P}^*$, whence $\mathscr{T}(P) \leqslant \mathscr{T}$. But then $\mathscr{T}(P) = \mathscr{T}$, whereas $O_2(\langle \mathscr{T} \rangle) = 1$ a contradiction.

(3.7) COROLLARY. *$C_T(D) = 1$ for all $T \in \mathscr{T}$.*

Proof. Since T admits $C_G(D)$, we have $C_T(D) \leqslant O_2(H) \leqslant \bigcap_{T\in\mathscr{T}} T = 1$.

(3.8) LEMMA. *Let $d \in D^{\#}$ and let $T \in \mathscr{T}$, with $C_T(d) \neq 1$. Then $C_T(d) \in \mathscr{T}_d$.*

Proof. Suppose false. Let Γ be the set of all ordered triples (R, S, T) such that $1 \neq R = C_T(d) \notin \mathscr{T}_d$, $\{S, T\} \leqslant \mathscr{T}$, and $R \leqslant C_S(d) \in \mathscr{T}_d$. Then Γ is non-empty. Choose $(R, S, T) \in \Gamma$ so that first $|R|$ and then $|S \cap T|$ is as large as possible. Set $Q = S \cap T$, and put $N = \langle \mathscr{T}(N(Q))\rangle$.

We note first that, by the maximality of R, whenever we have $R < C_{\hat{S}}(d)$ for some $\hat{S} \in \mathscr{T}$, then $C_{\hat{S}}(d) \in \mathscr{T}_d$. Suppose first that N is a 2-group, and let $N \leqslant S^* \in \mathscr{T}$. Then $S^* \cap T \geqslant N_T(Q) > Q$, and so $(R, S^*, T) \notin \Gamma$. Then $C_{S}^*(d) \notin \mathscr{T}_d$, and so $C_{S}^*(d) = R$. But then $(R, S, S^*) \in \Gamma$, whereas $S \cap S^* \geqslant N_S(Q) > Q$; a contradiction. Thus, N is not a 2-group, and so $N \in \mathscr{P}$. We fix $P \in \mathscr{P}^*$ with $N \leqslant P$.

Let $N_S(Q) \leqslant S_1 \in \mathscr{T}(P)$, $N_T(Q) \leqslant T_1 \in \mathscr{T}(P)$. Thus $S_1 \neq T_1$, and $\{S_1, T_1\} = \mathscr{T}(P)$. Since $S \cap S_1 \geqslant N_S(Q) > Q$, we have $(C_{S_1}(d), S, S_1) \notin \Gamma$ and so $C_{S_1}(d) \in \mathscr{T}_d$. Also, $T \cap T_1 \geqslant N_T(Q) > Q$, so $(R, T_1, T) \notin \Gamma$, which implies that $C_{T_1}(d) \notin \mathscr{T}_d$, and then $C_{T_1}(d) = R$. Hence $(R, S_1, T_1) \in \Gamma$, and $S_1 \cap T_1 = Q = O_2(P)$, so that we may take $S = S_1$ and $T = T_1$. As $R < C_S(d)$, $C_S(d) \not\leqslant Q$, and so d does not induce a non-identity inner automorphism on P/Q. Hence $C_P(d)Q \in \mathscr{P}$, and $W \cap (C_P(d)Q)$ fixes $\langle d\rangle$ and fuses R to $C_S(d)$, whereas $|R| \neq |C_S(d)|$; a contradiction.

(3.9) LEMMA. *Let $d \in D^{\#}$, and suppose $\mathscr{P}_d \neq \varnothing$. Then $C_G(d)$ satisfies our Hypothesis, with $\mathscr{T}_d$ and $\mathscr{P}_d$ in place of $\mathscr{T}$ and $\mathscr{P}$.*

Proof. Let P_0 be maximal (under inclusion) in $\mathscr{P}_d$, let $S \in \mathscr{T}(P_0)$, and let $P_0 \leqslant P_1 \in \mathscr{P}^*$, with $S \leqslant T \in \mathscr{T}(P_1)$. Then $\langle C_T(d), P_0\rangle \in \mathscr{P}_d$, so that $S = C_T(d)$. Thus $S \in \mathscr{T}_d$, by (3.8). Since W is transitive on $\mathscr{T}$, by (3.4), it follows that every member of $\mathscr{T}_d$ is conjugate to S via W, and then $W_{\langle d\rangle}$ is transitive on $\mathscr{T}_d$, by (3.7).

Since $\mathscr{T}$ is complete in $\mathit{И}_G^*(H; 2)$, $\mathscr{T}_d$ is complete in $\mathit{И}_{C(d)}^*(H; 2)$. Let $\mathscr{S}$ be the smallest complete subset of $\mathscr{T}_d$ containing S. We will show that $\mathscr{S} = \mathscr{T}_d$. Namely, by (3.2), it will suffice to show that whenever $R \in \mathscr{S}$ and $R_1 \in \mathscr{T}_d$ with $\langle R, R_1\rangle \leqslant P$ for some $P \in \mathscr{P}^*$, then $R \cap R_1 \neq 1$. Assume by way of contradiction that $R \cap R_1 = 1$. Then $R \cap O_2(P) = 1$, so that by (3.1), d

either centralizes or induces a field automorphism on $P/O_2(P)$. In either case, $C_P(d)$ is a Bender group, with Cartan subgroup $H \cap C_P(d)$ and with Sylow 2-subgroup R. Then H acts irreducibly on R/R' and on $Z(R)$. Since R is conjugate to S via $W_{\langle d \rangle}$, we have $R \leqslant P_2$ for some $P_2 \in \mathscr{P}_d$, and $R \in \mathscr{T}(P_2)$. Then $Z(R) = O_2(P_2) \leqslant Z(P_2)$, and since $D = \langle D \cap P_2, d \rangle$, we have $Z(R) \leqslant C(D)$. This contradicts (3.7), so we conclude that $R \cap R_1 \neq 1$, so that $\mathscr{S} = \mathscr{T}_d$. Thus $\mathscr{T}_d$ is a component of $\mathcal{U}^*_{C(d)}(H; 2)$.

We have already shown that $\mathscr{T}(P_0) \leqslant \mathscr{T}_d$ for any $P_0 \in \mathscr{P}^*$, so it only remains to show that $O_2(\langle \mathscr{T}_d \rangle) = 1$. Suppose false, and put $Q = O_2(\langle \mathscr{T}_d \rangle)$. We claim that $Q \leqslant \bigcap_{T \in \mathscr{T}} T$. Suppose false. Then by (3.2), there exists $P^* \in \mathscr{P}^*$ with $Q \leqslant P^*$ and with $Q \nleqslant O_2(P^*)$. But then $C_{P^*}(d)/O_2(C_{P^*}(d))$ is a Bender group, so that $Q \ntrianglelefteq C_{P^*}(d)$. Since $C_{P^*}(d) \leqslant \langle \mathscr{T}_d \rangle$, we have a contradiction. Thus $Q \leqslant \bigcap_{T \in \mathscr{T}} T$, as claimed. But then $Q = 1$, by (3.6). This completes the proof of (3.9)

4. Lattices

In this section, we fix $(P_0, P_1) \in \Delta$. Let T_0 be the unique member of $\mathscr{T}(P_0) \cap \mathscr{T}(P_1)$. By (3.1) there exist involutions $x_1 \in W \cap P_1$ and $x_{-1} \in W \cap P_0$. Let I denote the integers.

For $i \in I$, set

$$x_{2i} = (x_{-1}x_1)^i, x_{2i+1} = x_1 x_{2i} \quad \text{for} \quad i \geqslant 0, \quad \text{and} \quad x_{2i-1} = x_{-1}x_{2i} \quad \text{for} \quad i \geqslant 0. \quad (*)$$

Thus $x_0 = 1$, $x_2 = x_{-1}x_1$, $x_3 = x_1x_{-1}x_1$, $x_{-2} = x_1x_{-1}$, $x_{-3} = x_{-1}x_1x_{-1}$, and so on. Notice that

(4.1) Lemma. *$\langle x_{-1}, x_1 \rangle$ is dihedral. The set $\{x_{2i+1} : i \in I\}$ consists of involutions, the set $\{x_{2i} : i \in I\}$ is a maximal cyclic subgroup of $\langle x_{-1}, x_1 \rangle$, and $x_{2i}^{-1} = x_{-2i}$ for all i. If $x_{2m} = 1$, then $|\langle x_{-1}, x_1 \rangle|$ divides $2m$.*

(4.2) Lemma. *We have $x_i x_j = x_{i+j}$ if j is even, and $x_i x_j = x_{j-i}$ if j is odd.*

Proof. This follows directly from the construction (*).

We fix the following notation. For all $i \in I$, set $T_i = T_{i,i} = T_0^{x_i}$, and set $Z_i = Z_{i,i} = \Omega_1(Z(T_i))$. For all pairs of indices i and j with $i < j$, define $T_{i,j}$ and $Z_{i,j}$ inductively, by

$$T_{i,j} = T_{i,j-1} \cap T_j, \qquad Z_{i,j} = Z_{i,j-1} Z_j.$$

We also set $P_i = \langle T_{i-1}, T_i \rangle$ and $U_i = \Omega_1(Z(P_i))$, and note that this is consistent with our original P_0 and P_1.

(4.3) Lemma. *The following hold for all i, j, k in I, with $i \leqslant j$.*

$$\text{(i)}\quad (T_{i,j})^{x_k} = \begin{cases} T_{i+k,j+k} \text{ if } k \text{ is even} \\ T_{k-j,k-i} \text{ if } k \text{ is odd} \end{cases}$$

$$\text{(ii)}\quad (P_i)^{x_k} = \begin{cases} P_{i+k} \text{ if } k \text{ is even} \\ P_{k-i+1} \text{ if } k \text{ is odd} \end{cases}$$

Proof. Immediate from (4.2). Notice that similar fomulas hold for

$$(Z_{i,j})^{x_k} \quad \text{and} \quad (U_i)^{x_k}.$$

(4.4) LEMMA. *$x_{2i-1} \in W \cap P_i$ for all i.*

Proof. This is by definition when i is 0 or 1. Now $x_{-2j}x_1x_{2j} = x_{4j+1}$ and $x_{-2j}x_{-1}x_{2j} = x_{4j-1}$, so that

$$x_{4j+1} \in (W \cap P_1)^{x_{2j}} = W \cap P_{2j+1}, \quad \text{and}$$
$$x_{4j-1} \in (W \cap P_0)^{x_{2j}} = W \cap P_{2j},$$

for any j. This yields (4.4).

(4.5) LEMMA. *We have $(P_i, P_{i+1}) \in \Delta$ for all i, and $\langle x_{2i-1}, x_{2i+1}\rangle = \langle x_{-1}, x_1\rangle$.*

Proof. Suppose $P_1 = P_2$. Then $x_3 \in W \cap P_1$, by (4.4), and then as $x_3 = x_1x_2$, we have $x_2 \in W \cap P_1$. But $x_2 = x_{-1}x_1$, so that $x_{-1} \in W \cap P_1$, and then $P_0 = \langle T_0, T_0^{x^{-1}}\rangle = \langle T_0, T_1\rangle = P_1$; a contradiction. Hence $P_1 \neq P_2$, and since $T_1 \in \mathscr{T}(P_1) \cap \mathscr{T}(P_2)$, we have $(P_1, P_2) \in \Delta$. If i is even then the ordered pair (P_0, P_1) is conjugate to (P_i, P_{i+1}) via x_i, and (P_1, P_2) is conjugate to (P_i, P_{i+1}) via x_{i-1} if i is odd. This yields $(P_i, P_{i+1}) \in \Delta$ for all i. Moreover, we have shown that $\langle x_1, x_3\rangle = \langle x_{-1}, x_1\rangle$ so it follows that $\langle x_{2i-1}, x_{2i+1}\rangle = \langle x_{-1}, x_1\rangle$.

(4.6) LEMMA. *$O_2(\langle P_i, P_{i+1}\rangle) = 1$ for all i.*

Proof. Suppose false, and set $P = \langle P_i, P_{i+1}\rangle$, so that $P \in \mathscr{P}$. By (4.5), we may assume that $i = 0$. Thus $\langle x_{-1}, x_1\rangle \leqslant W \cap P$, and since $x_{-1} \neq x_1$, if follows from (3.1) that every element of $D - P$ induces a non-identity field automorphism on $P/O_2(P)$, and that $\langle x_{-1}, x_1\rangle = W \cap P$ is dihedral of order $2p$. Since $x_{-1} \notin W \cap P_1$, $C_D(P_1/O_2(P_1)) \neq 1$, and so $P_1 \leqslant O_2(P)\, C_P(d_1)$ for some $d_1 \in D^{\#}$. Similarly, $P_0 \leqslant O_2(P)\, C_P(d_0)$ for some $d_0 \in D^{\#}$. But $T_0 \leqslant P_0 \cap P_1$, so $T_0 \leqslant O_2(P)\, C_{T_0}(\langle d_0, d_1\rangle)$. By (3.7) it follows that $\langle d_0\rangle = \langle d_1\rangle$, so that $C_D(P/O_2(P)) \neq 1$; a contradiction.

(4.7) LEMMA. *$\bigcap_{i\in I} T_i = 1$.*

Proof. Suppose false, and set $Q = \bigcap_{i\in I} T_i$. Set $S_i = N_{T_i}(Q)$ for all i, and suppose first that $S_0 \nleqslant O_2(P_1)$. Since Q admits $\langle x_{-1}, x_1\rangle$, we then have

$S_1 = S_0^{x_1} \nleq O_2(P_1)$, so that $N_{P_1}(Q) \in \mathscr{P}$. Since $P_{-1}^{x_2} = P_1$, we also have $N_{P_{-1}}(Q) \in \mathscr{P}$. Set $P = \langle S_i : i \in I \rangle$. Thus $P \in \mathscr{P}$ and by (3.5), P is contained in a unique member P^* of $\mathscr{P}^*$. Then also P^* is the unique member of $\mathscr{P}^*$ containing both P_{-1} and P_1, so that $\langle P_{-1}, P_1 \rangle \in \mathscr{P}$. But $P_0 = \langle T_{-1}, T_0 \rangle \leqslant \langle P_{-1}, P_1 \rangle$, and so $\langle P_0, P_1 \rangle \in \mathscr{P}$; contrary to (4.6). We therefore conclude that $S_0 \leqslant O_2(P_1)$. A similar argument yields $S_0 \leqslant O_2(P_0)$.

Now $S_0 = N(Q) \cap O_2(P_1) = N(Q) \cap O_2(P_0)$ admits both x_1 and x_{-1}. Since $\langle x_{-1}, x_1 \rangle$ is transitive on $\{T_i : i \in I\}$, it follows that $S_0 = Q$, whence $Q = T_0$, which is absurd. This proves (4.7).

(4.8) LEMMA. *For all $i \in I$, we have $T_{i-1,i+1} = T_{i-1} \cap T_{i+1}$.*

Proof. Set $Q = T_{i-1,i+1}$. Then $Q = O_2(P_i) \cap O_2(P_{i+1})$, so that $Q \trianglelefteq T_i$. Set $R = T_{i-1} \cap T_{i+1}$, and suppose that $Q < R$. Then $Q < N_R(Q)$, so that $N_R(Q) \nleq T_i$. Since $R \leqslant T_{i-1}$, we then have $P_i = \langle N_{T_{i-1}}(Q), T_i \rangle$, so that $Q \trianglelefteq P_i$. Similarly, $Q \trianglelefteq P_{i+1}$, but we then contradict (4.6). Hence (4.8) holds.

(4.9) LEMMA. *For all i and j with $i < j$ we have $T_{i,j} \trianglelefteq \langle T_{i,j-1}, T_{i+1,j} \rangle$, and if $i < j - 1$ then $T_{i,j} \trianglelefteq T_{i+1,j-1}$.*

Proof. Since $T_{i-1,i} = O_2(P_i)$, (4.9) holds if $j - i = 1$. Since $T_{i-1,i+1} = O_2(P_i) \cap O_2(P_{i+1}) \trianglelefteq T_i$, we also have (4.9) if $j - i = 2$. For $j - i > 2$, we apply induction to obtain $T_{i,j} = T_{i,j-1} \cap T_{i+1,j} \trianglelefteq T_{i+1,j-2} \cap T_{i+2,j-1} = T_{i+1,j-1}$.

(4.10) LEMMA. *Let i and j be indices with $i < j$, and let $d \in D\#$ with $[P_i, d] \leqslant O_2(P_i)$ and $[P_j, d] \leqslant O_2(P_j)$. Assume $T_{i,j} \neq 1$. Then $W_d \leqslant \langle x_{-1}, x_1 \rangle \leqslant W_{\langle d \rangle}$, $W_d \cong D_{2p}$, and $| W_{\langle d \rangle} : W_d | \leqslant 2$. Moreover, if $j - i$ is odd, then $\langle x_{-1}, x_1 \rangle = W_d$.*

Proof. We may assume without loss that $i = 1$. Thus $\langle x_1, x_{2j-1} \rangle \leqslant W_d$. Set $Y = \langle x_1, x_{2j-1} \rangle$, and suppose first that $x_1 \neq x_{2j-1}$. Then Y is dihedral, and since W acts faithfully on D, we then have $Y = W_d \cong D_{2p}$; a dihedral group of order $2p$. Since W is generated by involutions, any dihedral subgroup of W containing Y lies in $W_{\langle d \rangle}$ and has order $2p$ or $4p$. In particular, this holds for $\langle x_{-1}, x_1 \rangle$. Now assume also that $j - 1$ is odd. Then $x_{-j} x_1 x_j = x_{2j+1}$, so that x_1 and x_{2j+1} are conjugate in $W_{\langle d \rangle}$. But $\langle x_{-1}, x_1 \rangle = \langle x_{2j-1}, x_{2j+1} \rangle$, by (4.5), so that $\langle x_{-1}, x_1 \rangle = W_d$ in this case.

We may now assume that $x_1 = x_{2j-1}$, so that $x_1 x_{2j-1} = x_{2j-2} = 1$. Then $|\langle x_{-1}, x_1 \rangle|$ divides $2j - 2$, by (4.1). Since $x_i x_{2j-2+i} = x_{2j-2}$ for all odd i, we have $x_i = x_{2j-2+i}$ for all odd i. Now $T_{1,j} \leqslant T_{j-1,j} = O_2(P_j)$, so that $\langle T_{1,j}, (T_{1,j})^{x_{2j-1}} \rangle$ is a 2-group. Hence $\langle T_{1,j}, (T_{1,j})^{x_1} \rangle$ is a 2-group, so that $T_{1,j} \leqslant O_2(P_1) \leqslant T_0$, and so $T_{1,j} = T_{0,j}$. Similarly, $T_{0,j-1} = T_{-1,j-1}$, and in fact $T_{1,j} = T_{0,j} \leqslant T_{0,j-1} = T_{-1,j-1} \leqslant T_{-1,j-2} = \cdots$, and so on, so that $T_{1,j} = 1$, by (4.7). But $T_{1,j} \neq 1$, by assumption; so $x_1 \neq x_{2j-1}$, and (4.10) is proved.

(4.11) THEOREM. *Suppose $O_2(Z(P_0)) = O_2(Z(P_1)) = 1$. Then each T_i is of type $L_3(2^n)$, for some fixed n.*

The proof of Theorem (4.11) will be carried out in a sequence of steps.

(4.12) *For any $i \in I$, we have $P_i/O_2(P_i)$ isomorphic to $L_2(2^n)$ for some n, $n > 1$, and $\Omega_1(Z(O_2(P_i)))$ is a natural module for $P_i/O_2(P_i)$.*

Proof. By (4.3), any P_i is conjugate to either P_0 or P_1. Now (5.1) follows from (4.5) and (2.7)

(4.13 $\Omega_1(Z(O_2(P_i))) = Z_{i-1}Z_i$.

Proof. Since $P_i = \langle T_{i-1}, T_i \rangle$, (4.13) is immediate from (4.12).

By (4.7), there exist indices $j \leqslant k$ with $Z_{j,k} \leqslant T_{j,k}$ and with $Z_{j,k+1} \nleqslant T_{j,k+1}$. Among all such pairs of indices, choose j and k with $k - j$ as small as possible. Then $k - j \geqslant 1$, by (4.13). In order to simplify our notation, we assume that $j = 1$. Thus $k - 1$ is 'minimal'.

Set $A = Z_{1,k}$, $T = T_k$, $P = P_{k+1}$, $x = x_{2k+1}$, $Q = O_2(P)$, $A_0 = A \cap Q$, and $K = \langle A, A^x \rangle$.

(4.14) *We have $A \leqslant P$, A is abelian, $A_0 = Z_{2,k}$, and $Z_1 \cap Q = 1$.*

Proof. Since $A \leqslant T_{1,k} \leqslant C(Z_{1,k})$, A is abelian. Since $T_{1,k} \leqslant T \leqslant P$, we have $A \leqslant P$. By the minimality of $k - 1$, we have $Z_{2,k+1} \leqslant T_{2,k+1} \leqslant T_{k,k+1} = Q$, so $Z_{2,k} \leqslant A_0$. Suppose $Z_1 \cap Q \neq 1$. We have $Z_{k+1} \leqslant T_{2,k+1} \leqslant T_2 \leqslant P_2$, so that Z_{k+1} acts on $Z_1Z_2 = Z(O_2(P_2))$. Since $Z_2 \leqslant C(Z_{k+1})$ and $Z_1 \cap Q \neq 1$, it follows that Z_{k+1} centralizes Z_1Z_2, whence $Z_1 \leqslant Q$ and $Z_{k+1} \leqslant O_2(P_2)$. Thus $Z_1 \leqslant T_{k+1}$ and $Z_{k+1} \leqslant T_1$, whence $Z_{1,k+1} \leqslant T_{1,k+1}$, contrary to our choice of A. We conclude that $Z_1 \cap Q = 1$, whence also $A_0 = Z_{2,k}$.

(4.15) *$KQ = P$, $K \in \mathscr{P}$, and $T_{1,2k} \leqslant C(K)$.*

Proof. Since $Z_1 \cap Q = 1$ and $|Z_1| = 2^n$, with $P/Q \simeq L_2(2^n)$, it follows that $KQ = P$. Since $k > 1$, $K \in \mathscr{P}$. Also, $C(A) \geqslant T_{1,k}$, so $C(K) \geqslant T_{1,k} \cap (T_{1,k})^x = T_{1,2k}$.

(4.16) $k = 2$.

Proof. Since Z_kZ_{k+1} is a natural module for P/Q, we have $[Z_1, Z_{k+1}] = Z_k$. But also $Z_{k+1} \leqslant Z_{2,k+1} \leqslant T_{2,k+1} \leqslant T_2 \leqslant P_2$, so that $[Z_1, Z_{k+1}] = Z_2$. Thus $Z_2 = Z_k$.

Suppose first that k is even, and set $y = x_{k+1}$. Then $Z_1 = Z_k{}^y = Z_2{}^y = Z_{k-1}$. Since $Z_{2,k} \leqslant A_0$ it follows that $k = 2$. So assume that k is odd. Since $Z_2 = Z_k$, we have $Z_0 = (Z_2)^{x_{-2}} = Z_{k-2}$. Since $|Z_0Z_1| = 2^{2n}$, $Z_0 \neq Z_1$, so $Z_1 \neq Z_{k-2}$ and $k > 3$. Hence $Z_{k-2} \leqslant Z_{2,k} \leqslant Q$, so that $Z_0 \leqslant Q$. Since $Z_{0,k-1} \leqslant T_{0,k-1}$, we then have $Z_0 \leqslant T_{0,k+1}$, and so $Z_1 = Z_0^{x_1} \leqslant T_{-k,1}$. Also, $Z_1 \leqslant T_{1,k}$, so $Z_1 \leqslant T_{-k,k}$. Setting $z = x_k$, we have $Z_{k-1} = Z_1{}^z \leqslant T_{0,2k} \leqslant C(K)$, by (4.15). Since $KQ = P$, $W \cap K = W \cap P$, and so Z_{k-1} is fixed by x. Thus $Z_{k-1} = Z_{k+2}$.

Set $U = Z_{k-1}$, and set $R = C_T(U)$. Then $R \geqslant \langle T_{k,k+2}, T_{k-1,k+1}\rangle$, and since $Q = T_{k,k+1}$ it follows that $R \leqslant Q$, and that $|Q/R| \leqslant 2^n$. Then $[K, Q] \leqslant R$, by (2.2), whence $KR \trianglelefteq P$. But then $Z(P) \geqslant Z(KR) \cap Z(Q) \neq 1$; a contradiction. This proves (4.16).

We now complete the proof of Theorem (4.11). Since $k = 2$, $P = P_3$, $Q = T_{2,3}$, and $T_{1,3} \leqslant C_Q(A)$. But $|T_{2,3} : T_{1,3}| \leqslant 2^n$, and so $|Q: C_Q(K)| \leqslant 2^{2n}$. Thus $Q = Z(Q)\, C_Q(K)$. Then $Q = Z(Q) \times C_Q(K)$, so that $C_Q(K) = 1$. Thus Q is a natural module for P/Q, and $T = Z_{1,3}$ is of type $L_3(2^n)$, by (2.9). Since $\langle x_{-1}, x_1\rangle$ is transitive on $\{T_1 : i \in I\}$, each T_i if of type $L_3(2^n)$.

5. Root-Strings

In this section, we begin to make use of the decomposition $S = \prod_{d \in D\#} C_S(d)$ for H-invariant 2-subgroups S of G (where the product is for some ordering of $D\#$).

(5.1) Theorem. *Suppose $C_D(T) \neq 1$ for some $T \in \mathscr{T}$. Let $d \in C_D(T)\#$. Then $\mathscr{T} = \mathscr{T}_d$ and $\langle \mathscr{T} \rangle$ is of type $L_3(2^n)$ for some $n > 1$. That is, part* (i) *of Theorem A holds.*

Proof. By (3.3), there exists $(P_0, P_1) \in \Delta^*$ with $\{T\} = \mathscr{T}(P_0) \cap \mathscr{T}(P_1)$. Then $D = \langle d, D \cap P_i\rangle$ for $i = 0$ and $i = 1$, so that $O_2(Z(P_i)) \leqslant C_T(D)$. But $C_T(D) = 1$, by (3.7). Hence $\langle P_0, P_1\rangle$ is of type $L_3(2^n)$ for some n, $n > 1$, by Theorem (4.13). Then $\mathscr{A}(T_i) = \{O_2(P_{i-1}), O_2(P_i)\}$ for all i in I, whence also $\mathscr{P}(T_i) = \{P_{i-1}, P_i\}$. It then follows from (3.2) and (4.5) that $\langle P_0, P_1\rangle = \langle \mathscr{T} \rangle$. Since d centralizes T, d centralizes both P_0 and P_1, and so $\langle \mathscr{T} \rangle \leqslant C(d)$, which proves (5.1).

(5.2) Lemma. *Let $d \in D\#$, with $\mathscr{T}_d \neq \{1\}$. Then either*:

(i) *For any $R \in \mathscr{T}_d$ there exist S in $\mathscr{T}_d$ and $P \in \mathscr{P}$ such that $\langle R, S\rangle = C_p(d)$ is a Bender group, or*

(ii) *$\langle \mathscr{T}_d \rangle$ is of type $L_3(2^n)$.*

Proof. Suppose first that $\mathscr{P}_d \neq \varnothing$. Then $C_G(d)$ satisfies our main Hypothesis, with $\mathcal{T}_d$ in place of $\mathcal{T}$, by (3.9), in which case (ii) follows from (5.1). So assume that $\mathcal{P}_d = \varnothing$, and let $R \in \mathcal{T}_d$. Since $\bigcap_{T \in \mathcal{T}} T = 1$, by (3.6), it follows from (3.2) that there exists $P \in \mathcal{P}^*$ with $R \leqslant P$ and with $R \nleqslant O_2(P)$. Then d does not induce a non-identity inner automorphism on $P/O_2(P)$, and since $\mathcal{P}_d = \varnothing$, it follows that $C_P(d)$ is a Bender group. Now $O_2(P)\, C_p(d) \in \mathcal{P}$, and then (3.1) yields that $C_P(d) = \langle R, R^x \rangle$ for some $x \in W \cap O_2(P)\, C_P(d)$. Then $x \in W_d$, whence $R^x \in \mathcal{T}_d$. Setting $S = R^x$, we have (i).

(5.3) LEMMA. *Let $d \in D\#$, and suppose that $\langle \mathcal{T}_d \rangle$ is of type $L_3(2^n)$. Then*

(i) *$C_T(d) \in \mathcal{T}_d$ for all $T \in \mathcal{T}$,*

(ii) *For any $S \in \mathcal{T}_d$ and any $A \in \mathcal{A}(S)$, $\langle \mathcal{T}(C(A)) \rangle \notin \mathcal{P}$, and*

(iii) *W_d is dihedral, of order $2p$.*

Proof. Let $R \in \mathcal{T}_d$. Then R is of type $L_3(2^n)$, and so if $R \leqslant P \in \mathcal{P}$, we have $R \cap O_2(P) \neq 1$. Then $C_T(d) \neq 1$, for any $T \in \mathcal{T}(P)$, and (i) follows now from (3.2) and (3.8).

Let $A \in \mathcal{A}(R)$. By (3.9) and (3.2) R is contained in at least two members of $\mathcal{P}_d$, so that $A = O_2(P)$ for some P in $\mathcal{P}_d$. If also $P_0 = \langle \mathcal{T}(C(A)) \rangle \in \mathcal{P}$, then $\langle P_0, P \rangle \in \mathcal{Y} - \mathcal{P}$, contrary to our Hypothesis. Hence (ii) holds.

We have $W_d = W \cap C(d)$. Since W operates faithfully on D and is generated by involutions, by (3.1), either (iii) holds or $| W_d | = 2$. But W_d acts transitively on $\mathcal{T}_d$, by (3.4) and (3.9), and $O_2(\langle \mathcal{T}_d \rangle) = 1$, by (3.6). Hence $\langle \mathcal{T}_d \rangle \notin \mathcal{P}_d$, and $| \mathcal{T}_d | > 2$, so that (iii) holds. This proves (5.3).

For any $d \in D\#$, set $\mathcal{R}_d = \{Z(T) \colon T \in \mathcal{T}_d\}$ if $\langle \mathcal{T}_d \rangle$ is of type $L_3(2^n)$, and otherwise set $\mathcal{R}_d = \mathcal{T}_d$. Set $\mathcal{R} = \bigcup_{d \in D\#} \mathcal{R}_d$ and set $\mathcal{Z} = \{Z(R) \colon R \in \mathcal{R}\}$. For any subgroup X of G, set $\mathcal{R}(X) = \{R \in \mathcal{R} \colon R \leqslant X\}$, $\mathcal{Z}(X) = \{Z \in \mathcal{Z} \colon Z \leqslant X\}$, and set $l(X) = | \mathcal{Z}(X)|$. For any $P \in \mathcal{P}^*$, set $L(P) = \langle \mathcal{R}(P) - \mathcal{R}(O_2(P)) \rangle$.

For $S \in \mathcal{U}_G(H; 2)$, we say that S is an *$\mathcal{R}$-string* if $C_S(d) = \langle \mathcal{R}(C_S(d)) \rangle$ for all $d \in D\#$ and that S is a *$\mathcal{Z}$-string* if $C_S(d) = \langle \mathcal{Z}(C_S(d)) \rangle$ for all $d \in D\#$. If S is either an $\mathcal{R}$-string or a $\mathcal{Z}$-string, we ray that S is a *root-string*.

(5.4) LEMMA. *Let $d \in D\#$ with $\langle \mathcal{T}_d \rangle$ of type $L_3(2^n)$, and let $T \in \mathcal{T}_d$. Then the following hold*:

(i) *$\mathcal{R}(T) = \mathcal{Z}(T)$,*

(ii) *$l(T) = 3$ and $T = \prod_{R \in \mathcal{R}(T)} R$ for any ordering of $\mathcal{R}(T)$, and*

(iii) *For any $A \in \mathcal{A}(T)$, A is a root-string, with $l(A) = 2$. Moreover, $| \mathcal{A}(T)| = 2$, and $\mathcal{A}(T) = \{O_2(P) \colon P \in \mathcal{P}_d(T)\}$.*

Proof. Part (i) is immediate from the definitions. Set $Z = Z(T)$, and let $X \in \mathcal{R}(T)$ with $X \neq Z$. Then $XZ \leqslant A$ for some $A \in \mathcal{A}(T)$. Let $X = Z(S)$, with $S \in \mathcal{T}_d$. Since $Z(P) = 1$ for any $P \in \mathcal{P}_d$, we have $\langle \mathcal{T}(C(X) \cap C(d)) \rangle = S$, whence $S \cap T \geqslant A$. Then $A \in \mathcal{A}(S)$, and as S and T are of type $L_3(2^n)$, we have $A \trianglelefteq \langle S, T\rangle$. Set $P = \langle S, T\rangle$. Then A is a natural module for P/A, whence X and Z are irreducible H-modules, by (2.8). Hence $A = XZ$, and since $|\mathscr{T}(P)| = 2$, we have $\{X, Z\} = \mathscr{R}(A)$, so that A is a root-string and $l(A) = 2$. We have $|\mathscr{P}_d(T)| \geqslant 2$, by (3.9) and (3.3). Since T is of type $L_3(2^n)$, we have $|\mathscr{A}(T)| = 2$, so in fact $|\mathscr{P}_d(T)| = 2$ and $\mathscr{A}(T) = \{O_2(N)\colon N \in \mathscr{P}_d(T)\}$. This also shows that X exists, and then that $l(T) = 3$. Since $T = AB$ for $\{A, B\} = \mathscr{A}(T)$, and since $A \cap B = Z$, T is the product of the members of $\mathscr{R}(T)$ in any order. Now all parts of the lemma are established.

(5.5) LEMMA. *Let $d \in D\#$ with $\langle \mathscr{T}_d\rangle$ of type $L_3(2^n)$, let S_1 and S_2 be root-strings, contained in $C_G(d)$. Then $S_1 \cap S_2$ is root-string.*

Proof. Assume false. Set $Q = S_1 \cap S_2$, $Z = \Omega_1(Z(Q))$, $R_i = N_{S_i}(Z)$ for $i = 1$ and 2, and $R = \langle R_1, R_2\rangle$. If $R \in \mathscr{P}$, then $O_2(R)$ is a minimal normal subgroup of R, whence $Z = O_2(R) \in \mathscr{A}(T)$ for any $T \in \mathscr{T}(R)$, where we have $\mathscr{T}(R) \leqslant \mathscr{T}_d$. Thus Z is a root-string in this case, by (5.4), and either $Z = Q$ or $Q \in \mathscr{T}(R)$, which implies that Q is a root-string. We conclude that R is a 2-group.

Since 1 is a root-string, neither S_1 nor S_2 is a minimal H-invariant 2-group, and so $l(S_i) \geqslant 2$ for $i = 1$ and 2, whence also $l(R_i) \geqslant 2$. Let $S_i \leqslant T_i \in \mathscr{T}_d$. Thus R_i contains some A_i in $\mathscr{A}(T_i)$, whence either $A_i = R_i$ or $R_i = T_i$. If $R_1 = T_1$, say, then as $R \notin \mathscr{P}$ we have $R = T_1$, and $Q = R_1 \cap R_2 = R_2$, whence $S_2 = Q$ is a root-string, which cannot be. Thus $R_i \neq T_i$ for $i = 1$ or 2, and so $A_i = R_i$. Since $R_1 \neq R_2$, this yields $l(R) = 3$, $R \in \mathscr{T}_d$, and $R_1 \cap R_2 = Q = Z(R)$; a root-string. This proves the lemma.

(5.6) LEMMA. *Let S be an $\mathscr{R}$-string (or $\mathscr{Z}$-string). Then $C_S(d)$ is an $\mathscr{R}$-string ($\mathscr{Z}$-string) for all $d \in D\#$, and*

$$S = \prod_{R \in \mathscr{R}(S)} R \quad \left(\text{or } S = \prod_{Z \in \mathscr{Z}(S)} Z\right)$$

for some ordering of $\mathscr{R}(S)$ or $\mathscr{Q}(S)$.

Proof. Since $S = \prod_{d \in D\#} C_S(d)$ for some ordering of $D\#$, we may assume that $S = C_S(d)$. If $S \in \mathscr{R}$ (or if $S \in \mathscr{Z}$) there is nothing to show. If $S \notin \mathscr{R} (S \notin \mathscr{Z})$ then it follows that $\langle \mathscr{T}_d\rangle$ is of type $L_3(2^n)$. Then (5.4) (ii) completes the proof of the lemma.

(5.7) LEMMA. *Let S_1 and S_2 be $\mathscr{R}$-strings ($\mathscr{Z}$-strings). Then $S_1 \cap S_2$ is an $\mathscr{R}$-string ($\mathscr{Z}$-string).*

Proof. Set $Q = S_1 \cap S_2$. Then $Q = \langle C_Q(d)\colon d \in D\#\rangle$, so we need only show that $\langle \mathscr{R}(C_Q(d)\rangle = C_Q(d)$ (or that $\langle \mathscr{Z}(C_Q(d))\rangle = C_Q(d)$) for all $d \in D\#$. Since $C_Q(d) = C_{S_1}(d) \cap C_{S_2}(d)$, we may assume that $\langle S_1, S_2\rangle \leqslant C(d)$. If

$\mathscr{R} \geqslant \mathscr{T}_d$ then either $S_1 = S_2$ or $S_1 \cap S_2 = 1$, by (3.7), so we may assume $\mathscr{R} \not\geqslant \mathscr{T}_d$. Then $\langle \mathscr{T}_d \rangle$ is of type $L_3(2^n)$, and an appeal to (5.5) completes the proof of (6.4).

(5.8) LEMMA. *Let $T \in \mathscr{T}$, $P \in \mathscr{P}^*$. Then T is an $\mathscr{R}$-string, $Z(T)$ is a $\mathscr{Z}$-string, and $O_2(Z(P))$ is a $\mathscr{Z}$-string with $l(O_2(Z(P))) \leqslant 1$.*

Proof. Let $d \in D\#$. Then either $C_T(d) = 1$ or $C_T(d) \in \mathscr{T}_d$, by (3.8). In either case, $C_T(d) = \langle \mathscr{R}(C_T(d)) \rangle$. Hence T is an $\mathscr{R}$-string. Set $Z = h(T)$. Then $C_Z(d) \leqslant Z(C_T(d))$. If $Z(C_T(d)) \neq 1$ then $Z(C_T(d))$ is an irreducible H-module, and so if $C_Z(d) \neq 1$ then $C_Z(d) \in \mathscr{Z}$. Hence Z is a $\mathscr{Z}$-string. We may assume that $\mathscr{T}(P) = \{S, T\}$, for some $S \in \mathscr{T}$, whence $O_2(Z(P)) = Z(S) \cap Z(T)$ is a $\mathscr{Z}$-string, by (5.7). Take $d \in D \cap P\#$. Then $O_2(Z(P)) \leqslant C(d)$, so that either $O_2(Z(P)) = 1$, $O_2(Z(P)) \in \mathscr{Z}$, or $\langle \mathscr{T}_d \rangle$ is of type $L_3(2^n)$ and $O_2(Z(P)) \in \mathscr{A}(C_T(d))$. The last possibility is ruled out by (5.3), so $l(O_2(Z(P))) \leqslant 1$.

(5.9) LEMMA. *Let $(P_0, P_1) \in \Delta^*$. Then, in the notation of section four, $T_{i,j}$ is an $\mathscr{R}$-string for all indices $i \leqslant j$, and $Z_{i,j}$ is a $\mathscr{Z}$-string if $Z_{i,j}$ is abelian.*

Proof. That $T_{i,j}$ is an $\mathscr{R}$-string follows immediately from (5.7) and (5.8). Let $d \in D\#$. Since each Z_k is a $\mathscr{Z}$-string, by (5.8), and since $Z_{i,j}$ is abelian, $C_{Z_{i,j}}(d)$ is a product of members of $\mathscr{Z}$, and is therefore a $\mathscr{Z}$-string. Hence $Z_{i,j}$ is a $\mathscr{Z}$-string.

(5.10) LEMMA. *Let S be an $\mathscr{R}$-string. Then $l(S) = |\mathscr{R}(S)|$.*

Proof. Recall that by definition, $l(S) = |\mathscr{Z}(S)|$. Let $Z \in \mathscr{Z}(S)$, and let $d \in C_D(Z)\#$. Then $C_S(d)$ is an $\mathscr{R}$-string, by (5.6). If $C_S(d) \in \mathscr{R}$ then $C_Z(d) = Z(C_S(d))$, and if $C_S(d) \notin \mathscr{R}$ then $\langle \mathscr{T}_d \rangle$ is of type $L_3(2^n)$ and $\mathscr{Z}(C_S(d)) = \mathscr{R}(C_S(d))$ by (5.3). This shows that $\mathscr{Z}(S) = \{Z(R) : R \in \mathscr{R}(S)\}$. Since $C_S(D) = 1$, by (3.7), distinct members of $\mathscr{R}(S)$ have distinct centers, so $|\mathscr{R}(S)| = |\mathscr{Z}(S)| = l(S)$.

At this point, we need the following refinement of (3.1).

(5.11) LEMMA. *Let $P \in \mathscr{P}$. Then $C_D(P/O_2(P)) \neq 1$.*

Proof. Assume false. Then there exists $P \in \mathscr{P}^*$ with $C_D(P/O_2(P)) = 1$. Let $d_0 \in D \cap P\#$, and let d_i, $1 \leqslant i \leqslant p$ be a set of generators for the set of cyclic subgroups of D not contained in P. Fix $T \in \mathscr{T}(P)$, set $T_i = C_T(d_i)$, $0 \leqslant i \leqslant p$, and set $Q = O_2(P)$.

Suppose first that $C_P(d) \notin \mathscr{P}$ for all $d \in D - P$. It follows then that $C_Q(d_i) = 1$ for all i, $1 \leqslant i \leqslant p$, so that $Q = T_0$. Then as $T_i \leqslant O^2(P)$, $i \leqslant i \leqslant p$, Q centralizes each such T_i. By (3.3) there is $P^* \in \mathscr{P}(T) - \{P\}$. Set $Q^* = O_2(P^*)$ and $Q_0^* = C_{Q^*}(d_0)$. Then $Q_0^* \leqslant Q$, while $T_i \leqslant C(Q)$ for $i > 0$. Thus $P^* =$

$\langle W \cap P^* \cap N(\langle d_0 \rangle), T\rangle \leqslant N(Q_0^*)$. As $P = C_p(Q)Q \leqslant N(Q_0^*)$ we conclude that $Q_0^* = 1$. Thus $Q^* \leqslant \langle T_1, ..., T_p\rangle \leqslant C(Q)$, so that $O^2(P^*)$ centralizes Q^*. Then also $d_0 \notin P^*$. Thus we may assume that $d_1 \in D \cap P^*$. Moreover, since $C_{Q^*}(D) = 1$, and since $T_i \neq 1$ for all i, it follows that $C_D(P^*/Q^*) = 1$.

By (3.1), there exists $x \in W \cap P$ and $y \in W \cap P^*$ with $|x| = |y| = p$. Moreover, $x \in W_{d_0}$ and $y \in W_{d_1}$, so that $x \neq y$. Thus $\langle x, y\rangle \cong SL(2, p)$, and so $|W: \langle x, y\rangle| = 2$, as W is generated by involutions. But $\langle x, y\rangle$ fixes T, and so $|\mathscr{T}| = 2$, by (3.4). But then $P = P^*$; a contradiction.

We conclude that $C_P(d_i) \in \mathscr{P}$ for some i, $1 \leqslant i \leqslant p$. Since $W \cap P$ is of order $2p$, we then have $C_P(d_i) \in \mathscr{P}$ for all i, $1 \leqslant i \leqslant p$. Then $\langle \mathscr{T}_d\rangle$ is of type $L_3(2^n)$ for all $d \in D - P$, by (3.9) and (5.2). It then follows from (5.3) that W contains two distinct subgroups of order p, whence W is transitive on $D\#$. Thus $\langle \mathscr{T}_{d_0}\rangle$ is of type $L_3(2^n)$, whence $T_0 \in \mathscr{T}_{d_0}$ by (5.3). Let $y \in W \cap P$, with y of order p. Then $y \in W_{d_0}$, and y fixes T_0. Since $|W_{d_0}| = 2p$, by (5.3), and since groups of type $L_3(2^n)$ satisfy our Hypotheses, we then have W_{d_0} intransitive on $\mathscr{T}_{d_0}$, contrary to (3.4). This completes the proof of (5.11).

Recall that for $P \in \mathscr{P}^*$ we have defined $L(P) = \langle \mathscr{R}(P) - \mathscr{R}(O_2(P))\rangle$.

(5.12) Corollary. *Let $P \in \mathscr{P}^*$. Then $|W \cap P| = 2$, $|\mathscr{R}(L(P))| = 2$, and $P = O_2(P)L(P)$, with $L(P) \cap O_2(P) = 1$.*

Proof. By (3.1) and (5.11), $|W \cap P| = 2$, and $C_D(P/O_2(P) \neq 1$. Let $d \in C_D(P/O_2(P))\#$. Then $L(P) \leqslant C_P(d)$, and $P = O_2(P)L(P)$. Suppose $L(P) \cap O_2(P) \neq 1$. Then $C_P(d) \in \mathscr{P}$, so that $\langle \mathscr{T}_d\rangle$ is of type $L_3(2^n)$, by (3.9) and (5.2). But then $L(P) = L(C_P(d))$ is a Bender group, by (5.4); a contradiction. Hence $L(P) \cap O_2(P) = 1$. Evidently $|\mathscr{R}(L(P))| = |\mathscr{T}(P)| = 2$, so (5.12) holds.

(5.13) Corollary. *Let $P \in \mathscr{P}^*$, $T \in \mathscr{T}(P)$, $Q = O_2(P)$. Then $l(T) - l(Q) = 1$. Also, if $(P_0, P_1) \in \Delta^*$ and $r \leqslant s$, then $l(T_{r,s}) = l(T_{r,s+1})$ or $l(T_{r,s}) = l(T_{r,s+1}) + 1$, and $l(T_{r,s}) = l(T_{r-1,s})$ or $l(T_{r,s}) = l(T_{r-1,s}) + 1$.*

Proof. The first statement is immediate from (5.12). Suppose $T_{r,s} \neq T_{r,s+1}$. Then $T_{r,s} \nleqslant T_{s,s+1}$, so that $T_{r,s}T_{s,s+1} = T_s$ and then $T_{r,s}/T_{r,s+1} \cong T_s/T_{s,s+1}$ so that $l(T_{r,s}) = l(T_{r,s+1}) + 1$. Similarly, if $T_{r,s} \neq T_{r-1,s}$ then $l(T_{r,s}) = l(T_{r-1,s}) + 1$.

(5.14) Lemma. *Let $(P_0, P_1) \in \Delta^*$, and let $l = l(T)$ for $T \in \mathscr{T}$. Then for any pair of indices i and j with $i \leqslant j$ we have $l(T_{i,j}) \geqslant l - j + i$. Moreover, equality holds if and only if $T_{i,k} > T_{i,k+1}$ for all k, $i \leqslant k < j$.*

Proof. By induction on $j - i$, we may assume that $i < j$ and that $l(T_{i,j-1}) = l - j + i + 1$. If $T_{i,j-1} = T_{i,j}$, there is then nothing to prove. So assume that $T_{i,j-1} > T_{i,j}$. That is, $T_{i,j-1} \nleqslant T_j$. Since $T_{i,j-1} \leqslant T_{j-1}$, this yields $T_{i,j-1} \leqslant P_j$ and $T_{i,j-1} \nleqslant O_2(P_j)$. Since $T_{i,j-1}$ and $T_{i,j}$ are $\mathscr{R}$-strings, by (5.9), it then follows from (5.10) that $l(T_{i,j-1}) - l(T_{i,j}) = 1$, which proves the lemma.

(5.15) LEMMA. *Let* $P \in \mathscr{P}^*$, $L = L(P)$. *Then for any* $R \in \mathscr{T}(C(L))$ *we have* $l(R) \leqslant 1$.

Proof. Let $d \in D \cap L\#$. *Then* $R \leqslant C(d)$, so if $l(R) > 1$ we have $\langle \mathscr{T}_d \rangle$ of type $L_3(2^n)$. But $\langle \mathscr{T}(C(R)) \rangle \in \mathscr{P}$, so $l(R) \leqslant 1$ by (5.3) and (5.4).

(5.16) LEMMA. *Let* S *be an* $\mathscr{R}$*-string,* $S \in \mathscr{T}(P)$, $P \in \mathscr{P}$, *with* $P/O_2(P) \cong L_2(2^n)$, *and with* $O_2(P)$ *a natural module for* $P/O_2(P)$. *Then* $l(S) = 3$, *and* $\mathscr{R}(S) = \mathscr{X}(S)$.

Proof. Let $\{S, T\} = \mathscr{T}(P)$. Then $T = S^x$ for some $x \in W \cap P$, so that T is an $\mathscr{R}$-string, and then so is $S \cap T$, by (5.7). Since $S \cap T = O_2(P)$ is natural module for $P/O_2(P)$, $S \cap T = Z(S)\,Z(T)$, where H acts irreducibly on $Z(S)$ and $Z(T)$. Hence $l(S \cap T) = 2$, and $\mathscr{R}(S \cap T) = \mathscr{X}(S \cap T)$. Let $R \in \mathscr{R}(S) - \mathscr{R}(T)$. Since H acts irreducibly on $S/O_2(P)$, we have $S = O_2(P)\,R$, and since $\mathscr{T}(C(D)) = \{1\}$, by (3.7), $\{R\} = \mathscr{R}(S) - R(T)$. Since $R \cap T$ is an $\mathscr{R}$-string, $R \cap T = 1$, so R is abelian, $R \in \mathscr{X}$, and $l(S) = 3$ with $\mathscr{R}(S) = \mathscr{X}(S)$.

6. THE STRUCTURE OF Δ^*

(6.1) LEMMA. *Let* $P \in \mathscr{P}^*$, $T \in \mathscr{T}(P)$. *Assume that* $F^*(P) = O_2(P)$ *and that no non-identity characteristic subgroup of* $J(T)$ *is normal in* P. *Then the following hold*:

(i) $P/O_2(P) \cong L_2(2^n)$, *for some* n, $n > 1$, *and* $O_2(P)/\Omega_1(Z(P))$ *is a natural module for* $P/O_2(P)$,

(ii) $\Phi(O_2(P)) = 1$ *and* $J(T) = T$,

(iii) $C_{O_2(P)}(D \cap P) = Z(P)$, *and*

(iv) $l(Z(P)) \leqslant 1$ *and* $l(T) = 3 + l(Z(P))$.

Proof. Set $Z = \Omega_1(Z(T))$, $Q = O_2(P)$, $V = \Omega_1(Z(Q))$, $S = J(T)$, and $L = L(P)$. By (2.6) we have $P/Q = L_2(2^n)$, P has just one-non-central 2-chief factor, and $[S, S] \leqslant Z(S)$. By (3.1), $n > 1$, and by (2.1), $V/\Omega_1(Z(P))$ is a natural module for P/Q.

Let $d \in D \cap P\#$. Notice that it follows from (5.11) and (3.1) that $|H \cap P| = 2^n - 1$, so that $V/\Omega_1(Z(P))$ is a sum of two irreducible H-submodules. Suppose first that $C_Q(d) = 1$. Then $V = Q$ is a natural module for P/Q, $T = J(T) = S$, and it follows from (5.7), (5.8) and (5.13) that $l(Q) = 2$ and $l(T) = 3$. Thus, (6.1) holds in this case. So assume that $C_Q(d) \neq 1$, and set $R = C_Q(d)$. Then $R \in \mathscr{T}_d$, by (3.8). Then either $R \in \mathscr{R}$ or R is of type $L_3(2^m)$ for some m, $m > 1$, and $l(R) = 3$. We have $O^2(P) \leqslant LV$, so $[L, R] \leqslant V$. Since $[L, R] \subseteq V \cap R \subseteq Z(R)$, it follows that $W \cap L$ fixes $\mathscr{R}(R)$ point-wise. Assume R is of type $L_3(2^m)$, and let $A \in \mathscr{A}(R)$. Then $W \cap L$ fixes A and $A \leqslant Q = VR$. Choose $P^* \in \mathscr{P}^*$ with $\langle \mathscr{T}(N(A)) \rangle \leqslant P^*$, and let $Q \leqslant T^* \in \mathscr{T}(P^*)$. Since $l(T) - l(Q) = 1$

and $l(T) = l(T^*)$, and since $\mathscr{R}(T) - \mathscr{R}(Q) \leqslant \mathscr{X}$, we have $Q \trianglelefteq T^*$, whence $T^* \in \mathscr{T}(P)$ and $P = \langle (T^*)^{W \cap L} \rangle$. But since $\langle \mathscr{T}(N(A)) \rangle \in \mathscr{P}$, P^* is the unique member of $\mathscr{P}^*$ containing $\langle \mathscr{T}(N(A)) \rangle$, by (3.5), so that P^* admits $W \cap L$. Thus $P^* = P$, whereas $R \nleqslant O_2(P^*)$, by (5.4); a contradiction. Hence $R \in \mathscr{R}$.

Suppose next that $R \notin \mathscr{X}$. Thus $\mathscr{R}(P) = \mathscr{X}(P) \cup \{R\}$, so that $R^W \cap \mathscr{R}(P) = \{R\}$. Let $P_1 \in \mathscr{P}(T) - \{P\}$. Then R does not admit $W \cap P_1$, by (4.7), and so $R \leqslant L(P_1)$. Set $U = Z(R)$. Then $U \leqslant V$, and as $U \leqslant C(d)$ we have $U \leqslant Z(P)$. Setting $K = \langle U^{W \cap P_1} \rangle$, it follows that K centralizes $O_2(P_1)$, whence also $[L(P_1), O_2(P_1)] = 1$, and then $l(O_2(P_1)) = 1$, by (5.15). Hence $l(T) = 2$, whereas we have $l(Q) = 3$; a contradiction.

Hence $R \in \mathscr{X}$, $\Phi(Q) = 1$, and $l(T) = 4$. Moreover, $Q \in \mathscr{A}(T)$, and since $J(T) \ntrianglelefteq P$ it follows that $T = QA$ for some $A \in \mathscr{A}(T) - \{Q\}$. Thus $T = S$, and all parts of (6.1) hold.

(6.2) Lemma. *We have* $|\mathscr{P}(T)| = 2$ *for all* $T \in \mathscr{T}$.

Proof. Suppose false, so that $|\mathscr{P}(T)| \geqslant 3$. We first show that if $P \in \mathscr{P}(T)$, then $Z(T) \leqslant O_2(P)$. For, set $Z = Z(T)$, and assume that $Z \nleqslant O_2(P)$. Then $\mathscr{X}(Z(T)) \nleqslant \mathscr{X}(O_2(P))$, whence it follows that $L(P)$ centralizes $O_2(P)$. Then also $O_2(P) \in \mathscr{T}_d$ for $d \in D \cap P\#$, by (3.8). For any $A \in \mathscr{A}(O_2(P))$ we have $C_P(A) \in \mathscr{P}$, so $\langle \mathscr{T}_d \rangle$ is not of type $L_3(2^n)$, by (5.3). Hence $O_2(P) \in \mathscr{R}$, and $l(T) = 2$. Then $Z = \langle \mathscr{X}(T) \rangle$ and $\mathscr{P}(T) = \{\langle \mathscr{T}(N(U)) \rangle : U \in \mathscr{X}(T)\}$, so that $|\mathscr{P}(T)| = 2$; a contradiction. Thus $Z \leqslant O_2(P)$, as clamed.

Now if T is of type $L_3(2^n)$ then $\mathscr{P}(T) = \{\langle \mathscr{T}(N(A)) \rangle : A \in \mathscr{A}(T)\}$ is of order two. Hence T is not of type $L_3(2^n)$. Then by (4.11) there do not exist distinct members P_0 and P_1 of $\mathscr{P}(T)$ with $O_2(Z(P_i)) = 1$. It then follows also that $Z \nleqslant Z(P)$ for any $P \in \mathscr{P}(T)$, as otherwise we may choose $P^* \in \mathscr{P}(T)$ with $P^* \neq P$ and with $O_2(Z(P^*)) \neq 1$; contrary to $\langle P, P^* \rangle \notin \mathscr{P}$.

Next suppose that $T = \langle \mathscr{T}(N(J(T))) \rangle$. Then $J(O_2(P)) < J(T)$ for all $P \in \mathscr{P}(T)$, whence $P/O_2(P) \cong L_2(2^n)$ for some n (depending on P) and $Z(O_2(P))/Z(P)$ is a natural module for $P/O_2(P)$, by (2.1). Also $l(Z(P)) \leqslant 1$, by (5.8), so $l(Z(T)) = 2$. Now for $P_i \in \mathscr{P}(T)$, $1 \leqslant i \leqslant 3$, $Z(P_i) = Z(P_j) \neq 1$ for some distinct i and j; a contradiction. We conclude that $\langle \mathscr{T}(N(J(T))) \rangle \in \mathscr{P}$.

Now choose $P \in \mathscr{P}(T)$ so that $J(T) \ntrianglelefteq P$. Then no non-identity characteristic group of $J(T)$ is normal in P, so that by (6.1), $\Phi(O_2(P)) = 1$, $l(T) = 3$ or 4, and $O_2(P)/Z(P)$ is a natural module for $P/O_2(P)$. It follows that $|\mathscr{A}(T)| = 2$, so that $|\mathscr{P}(T)| = 2$; a contradiction. This completes the proof of (6.2).

(6.3) Corollary. *Let* $(P_0, P_1) \in \Delta^*$. *Then* $\{P_i : i \in I\} = \mathscr{P}^*$, $\{T_i : i \in I\} = \mathscr{T}$, *and* $\langle x_{-1}, x_1 \rangle = W$ *is dihedral.*

Proof. Let $W_0 = \langle x_{-1}, x_1 \rangle$. Then $|\mathscr{P}(Tw) \cap (P_0^{W_0} \cup P_1^{W_0})| \geqslant 2$ for each $w \in W_0$, so by (6.2) and (3.2) we have $\{P_i : i \in I\} = \mathscr{P}^*$ and $\{T_i : i \in I\} = \mathscr{T}$.

Since $|P \cap W| = 2$ for all $P \in \mathscr{P}^*$, by (5.12), and since $|P \cap W_0| = 2$, we obtain $W = W_0$.

From now on, we fix $(P_0, P_1) \in \Delta^*$ and adopt the notation of section 4. Set $L_i = L(P_i)$ for all i.

(6.4) LEMMA. *Assume that there exist S and T in $\mathscr{T}$ with $S \cap T = 1$ and that $l(T) \geqslant \frac{1}{2}|\mathscr{T}|$. Then $l(T) = \frac{1}{2}|\mathscr{T}|$ and $\mathscr{R} = \mathscr{R}(S) \cup \mathscr{R}(T)$.*

Proof. We assume without loss that $T = T_0$, set $l = l(T)$. Set $m = |\mathscr{T}|$, so that $m \leqslant 2l$. It follows from (6.3) that $\mathscr{T} = \{T_0, T_1, \ldots, T_{m-1}\}$. For any i with $0 \leqslant i < l$ we have $l(T_{0,i}) \geqslant l - i \geqslant 1$, by (5.14), so that $T_0 \cap T_i \neq 1$. Also, for any j satisfying $l < j \leqslant m$, we have $T_0 \cap T_j = T_j \cap T_m \leqslant T_{j,m}$ and $l(T_{j,m}) \geqslant l - m + j \geqslant 1$, so that $T_0 \cap T_j \neq 1$. Hence $S = T_l$, and it follows by symmetry that $m = 2l$.

Suppose next that $T_{0,i} = T_{0,i+1}$ for some i with $0 \leqslant i < l$. Then $T_{0,l-1} = T_{0,l}$, whereas $l(T_{0,l-1}) \geqslant 1$; a contradiction. We therefore conclude that $T_{0,i} > T_{0,i+1}$ for all such i, whence $T \cap P_{i+1} \nleqslant O_2(P_{i+1})$, and so $T \cap L_{i+1} \neq 1$. Similarly, for all such i we have $T_{i,l} < T_{i+1,l}$, so that $S \cap L_{i+1} \neq 1$. Repeating this argument for all j with $l < j \leqslant 2l$, we see that $T \cap L_k \neq 1 \neq S \cap L_k$ for all $k \in I$. Hence $\mathscr{R}(L) = \{T \cap L, S \cap L\}$ for all $L = L_k$, $k \in I$, by (5.7) and (5.12). But $\mathscr{R} = \bigcup_I \mathscr{R}(L_k)$, by (4.7), so $\mathscr{R} = \mathscr{R}(S) \cup \mathscr{R}(T)$.

(6.5) LEMMA. *Assume that there exist S and T in $\mathscr{T}$ with $S \cap T = 1$ and with $l = l(T) \geqslant \frac{1}{2}|\mathscr{T}|$. Set $B = T(H \cap \langle\mathscr{T}\rangle)$ and $N = \langle N_P(D): P \in \mathscr{P}\rangle$. Then $\langle\mathscr{T}\rangle = \langle B, N\rangle$ is a split $B - N$-pair of rank two.*

Proof. Take $T = T_0$. We have $\langle\mathscr{T}\rangle = \langle B, N\rangle$ by (3.4), and $B \cap N = H \cap \langle\mathscr{T}\rangle$ by (3.7). By (6.3), $N/(H \cap \langle\mathscr{T}\rangle) \cong NH/H = W$ is generated by two involutions, x_1 and x_{-1}, neither of which normalizes B. In what follows we identify elements of W with cosets of $H \cap \langle\mathscr{T}\rangle$ in N when convenient. Thus, in order to complete the proof of (6.5) we have only to verify the Bruhat relations

$$x_i B x \leqslant BxB \cup Bx_i xB. \qquad (*)$$

for $i = \pm 1$ and for all $x \in W$. By symmetry it will in fact suffice to prove (*) in the case that $i = 1$.

We first observe that $B \cup Bx_1B$ is a subgroup of $\langle\mathscr{T}\rangle$. Namely, set $P = P_1$, $Q = O_2(P)$, $\{R\} = \mathscr{R}(T) - \mathscr{R}(Q)$, and $J = H \cap \langle\mathscr{T}\rangle$. Now Bx_1B is a union of $|B: B \cap B^{x_1}|$ distinct cosets of B in PJ, and then since $B \cap B^{x_1} = QJ$, $B \cup Bx_1B$ contains all $|R| + 1$ cosets of B in PJ. Thus $B \cup Bx_1B = PJ$, a subgroup of $\langle\mathscr{T}\rangle$.

Let $x \in W$, and suppose first that $R^x \leqslant T$. Then $x_1Bx = x_1JQRx \leqslant JQx_1xT \leqslant Bx_1xB$. In particular, (*) holds in this case. Now suppose that $R^x \nleqslant T$. Then $R^x \leqslant S$, by (6.4). Since L_1 is not a 2-group, and $L_1 = \langle R, R^{x_1}\rangle$, we have $R^{x_1x} \nleqslant S$, and so $R^{x_1x} \leqslant T$. Set $x' = x_1x$. Then $x_1Bx' \leqslant Bx_1x'B$, by the case just considered. Thus:

$$\begin{aligned} x_1Bx &\leqslant (Bx_1B)(Bx_1x'B) \leqslant (Bx_1B)(Bx_1B)(Bx'B) \\ &\leqslant (B \cup Bx_1B)(Bx'B) \qquad \text{(as } B \cup Bx_1B \text{ is a group)} \\ &\leqslant (Bx'B) \cup (Bx_1Bx'B) \leqslant (Bx'B) \cup (Bx_1x'B) \\ &= Bx_1xB \cup BxB, \end{aligned}$$

so that (*) holds in this case as well. This completes the proof of (6.5).

(6.6) LEMMA. *Let $d \in D\#$, with $\mathscr{T} \neq \mathscr{T}_d \neq \{1\}$, and with $|W_d| > 2$. Then $W = W_{\langle d \rangle}$ is dihedral of order $4p$, and $|W_{d'}| \leqslant 2$ for all $d' \in D - \langle d \rangle$.*

Proof. Since W is dihedral and acts faithfully on D, $|W_d| \mid 2p$. Since $\mathscr{T}_d \neq \{1\}$, it follows from (5.2) and (5.3) that $2 \mid |W_d|$, so W_d is dihedral of order $2p$. Suppose $W = W_d$. Since $\mathscr{T}_d \neq \mathscr{T}$, we have $[T_0, d] \neq 1$, by (5.1). Then by (4.7), there exists k, $k \geqslant 0$, with $[T_0, d] \leqslant T_{0,k}$ and $[T_0, d] \nleqslant T_{0,k+1}$. Setting $P = P_{k+1}$, it follows that $[T_0, d] \leqslant P$ and $[T_0, d] \nleqslant O_2(P)$. But $W \cap P \leqslant W_d$, so that $d \in C_D(P/O_2(P))$; a contradiction. Hence $W > W_d$. There exists only one dihedral subgroup of $SL^{\pm}(2, p)$ which properly contains a fixed subgroup of order $2p$; and it has order $4p$. Thus $|W| = 4p$, and $W = W_{\langle d \rangle}$.

(6.7) LEMMA. *Suppose $\mathscr{T}(L_i) \cap \mathscr{T}(L_j) \neq \varnothing$. Then either $L_i = L_j$, or $C_D(\langle \mathscr{T} \rangle) \neq 1$, or $j - i$ is even and $W = W_{\langle d \rangle}$ is dihedral of order $4p$ for $d \in C_D(L_i)^{\#}$.*

Proof. Assume $L_i \neq L_j$, and let $R \in \mathscr{T}(L_i) \cap \mathscr{T}(L_j)$. Then $R \in \mathscr{R}$, by (5.13), so that there exists $d \in C_D(R)^{\#}$. Then $\langle d \rangle = C_D(L_i) = C_D(L_j)$, by (3.7). Set $x = x_{2i-1}$, and $y = x_{2j-1}$. Thus $L_i = \langle R, R^x \rangle$ and $L_j = \langle R, R^y \rangle$, whence also $x \neq y$. Thus $\langle x, y \rangle$ is dihedral, and assuming $C_D(\langle \mathscr{T} \rangle) \neq \langle d \rangle$, $W = W_{\langle d \rangle}$ is of order $4p$, by (6.6). Suppose $j - i$ is odd. Then $L_{j+1} = (L_i)^{x_{j-i+1}} \leqslant C(d)$. But $W = \langle x_{-1}, x_1 \rangle = \langle x_{2j-1}, x_{2j+1} \rangle$, by (4.5), so that $W = W_d$, whereas $|W_d| = 2p$; a contradiction. Hence $j - i$ is even, which proves (6.7).

7. THEOREM A

We continue the notation of the last section. Thus $(P_0, P_1) \in \Delta^*$, and $L_i = L(P_i)$ for all i. Assume that G is a counter-example to Theorem A.

(7.1) LEMMA. Either $O_2(Z(P_0)) \neq 1$ *or* $O_2(Z(P_1)) \neq 1$.

Proof. Suppose false. Then $\langle P_0, P_1 \rangle$ is of type $L_3(2^n)$, for some n, by (4.11). Then also $\mathscr{R} = \mathscr{X}$ and $l(T) = 3$, by (5.16).

Set $A_i = O_2(P_i)$ for all i. Then $l(A_i) = 2$, $A_i A_{i+1} = T_i$, $\mathscr{A}(T_i) = \{A_i, A_{i+1}\}$, $A_i \cap A_{i+1} = Z_i = Z(T_i)$, and $T_i > A_i > Z_i > 1$ is a 2-chief series for $T_i H$. Also, we have $Z_i Z_{i+1} = A_{i+1}$, so that $Z_i \cap Z_{i+1} = 1$. Now $T_{0,1} = A_1$, $T_{0,2} = A_1 \cap A_2 = Z_1$, and $T_{0,3} = Z_1 \cap Z_2 = 1$. Similarly, $T_{-1,2} = 1$, and it follows that $Z_1 \leqslant L_0 \cap L_3$. Then $L_0 = L_3$, by (6.7), whence also $W \cap L_0 = W \cap L_3$. Thus $x_{-1} = x_5$, which yields $x_6 = 1$ and $|W| \leqslant 6$ by (4.1) and (4.2). Since W is transitive on $\mathscr{T}$, $|\mathscr{T}| \leqslant 6$, so that $l(T) \geqslant \frac{1}{2}|\mathscr{T}|$ for any $T \in \mathscr{T}$.

We have $T_0 = A_0 A_1 = Z_{-1,1}$ and $T_3 = A_3 A_4 = Z_{2,4}$. Suppose $T_0 \cap T_3 \neq 1$. Then by (5.7) we have $Z_i = Z_j$ for some i and j with $-1 \leqslant i < j \leqslant 4$. Since $P_2 = \langle T_1, T_2 \rangle$ we have $\mathscr{Z}(P_2) = \{Z_0, Z_1, Z_2, Z_3\}$ and $l(P_2) = 4$, and it is then clear that any four consecutive members of $\mathscr{Z}$ are distinct. Thus we must have $Z_{-1} = Z_4$. But $Z_{-1} = (Z_{-1})^{x_6} = Z_5 \neq Z_4$; a contradiction. We therefore conclude that $T_0 \cap T_3 = 1$. Now by (6.5), G is not a counter-example to Theorem A.

(7.2) LEMMA. *Suppose $W = W_{\langle d \rangle}$ for some $d \in D^{\#}$. Then either for all odd i or for all even i we have $d \in L_i$ and $L_{i+1} \leqslant C(d)$.*

Proof. By (4.7) and (6.3) we have $\mathscr{R} = \bigcup_I \mathscr{R}(L_i)$. Then $\langle L_i : i \in I \rangle \nleqslant C(d)$, by (5.1). Suppose $d \in L_i$ for all i. Then $\mathscr{R}(C(d)) = \varnothing$, so that $\langle \mathscr{T}(C(d)) \rangle = 1$ and $O_2(Z(P_i)) = 1$ for all i, contrary to (7.1). Thus $d \notin \bigcap_I L_i$. It follows that W_d is not the maximal cyclic subgroup $\langle x_2 \rangle$ of W. Also, by (5.1), $W \neq W_d$. Thus, if $x_{2i-1} \in W_d$, then since $W = \langle x_{2i-1}, x_{2i+1} \rangle$, by (4.5), we have $x_{2i+1} \in W - W_d$, and if $x_{2i-1} \in W - W_d$ then $x_{2i+1} \in W_d$. This proves the lemma.

(7.3) LEMMA. *$|W|$ is divisible by four.*

Proof. Let $P \in \mathscr{P}^*$, $L = L(P)$, $U = O_2(Z(P))$, and assume P chosen with $U \neq 1$. By (4.7) and (5.8), there exists $P^* \in \mathscr{P}^*$ with $U \leqslant L(P^*)$. Set $K = L(P^*)$, and take $x \in (W \cap L)^{\#}$. Thus $U \leqslant K \cap K^x$. If $K \neq K^x$ then $|W| = 4p$, by (6.7), so we may assume $K = K^x$. Then $[x, W \cap K] = 1$, and since $\langle U^{W \cap K} \rangle \neq U$ we have $x \notin W \cap K$, so that $|\langle x, W \cap K \rangle| = 4$.

(7.4) LEMMA. *W has precisely two orbits on $\mathscr{R}$.*

Proof. Since $W = \langle x_{-1}, x_1 \rangle$, it follows from (7.3) that x_{-1} and x_1 are non-conjugate in W. Hence L_0 and L_1 are not conjugate via W. Let $R \in \mathscr{R}(L_0)$, $S \in \mathscr{R}(L_1)$, and suppose that $R^x = S$ for some $x \in W$. Then $S \leqslant L_0^x \cap L_1$, where $L_0^x \neq L_1$, so that $\langle L_0^x, L_1 \rangle \leqslant C(d)$ for some $d \in D^{\#}$. Then $W = W_{\langle d \rangle}$, by (6.6), whence also $W = \langle x_{-1}, x_1 \rangle = W_d$, contrary to (7.3).

We conclude that R and S lie in distinct orbits. By (4.7), $\mathscr{R} = \bigcup_{i \in I} \mathscr{R}(L_i)$, and since every L_i is conjugate via W either to L_0 or to L_1, W has just two orbits on $\mathscr{R}$.

From now on, we denote the two orbits of W on $\mathscr{R}$ by $\mathscr{R}^+$ and $\mathscr{R}^-$, and the corresponding orbits on $\mathscr{L}$ by $\mathscr{L}^+$ and $\mathscr{L}^-$. For any $X \leqslant G$, set $\mathscr{R}^+(X) = \mathscr{R}(X) \cap \mathscr{R}^+$, $\mathscr{R}^-(X) = \mathscr{R}(X) \cap \mathscr{R}^-$, and similarly for $\mathscr{L}^+(X)$ and $\mathscr{L}^-(X)$. Set $l^+(X) = |\mathscr{L}^+(X)|$, $l^-(X) = |\mathscr{L}^-(X)|$, $l^+ = l^+(T)$, and $l^- = l^-(T)$, for $T \in \mathscr{T}$. Put $l = l^+ + l^-$, so that $l = l(T)$.

(7.5) LEMMA. *Let $R \in \mathscr{R}$, and suppose that $R \in \mathscr{T}_d$ for some $d \in D\#$. Then there exist indices r and s with $R = T_{r,s}$ and with $T_{r,s+1} = T_{r-1,s} = 1$. Moreover, $s - r + 1$ is even.*

Proof. By (4.7) there are indices r and s such that $R \leqslant T_{r,s}$, $R \nleqslant T_{r,s+1}$, and $R \nleqslant T_{r-1,s}$. Then $R \leqslant L_r \cap L_{s+1}$, so that $\langle L_r, L_{s+1}\rangle \leqslant C(d)$. Suppose first that $T_{r,s+1} \neq 1$. Then $W = W_{\langle d\rangle}$, by (4.10), so that in particular, for $x = x_2$, $R^x \leqslant C(d)$. By (7.2), $L_{r+1} \nleqslant C(d)$, so that $R \leqslant T_{r+1}$ and $s > r$. Also, $R^x \leqslant T_{r+2,s+2} \leqslant T_{s+1}$, so that $R \neq R^x$. Since $R \in \mathscr{T}_d$, it follows that $\langle R, R^x\rangle$ is not a 2-group, and so $R \nleqslant T_{r+2}$. Hence $s = r + 1$.

Without loss of generality, we may assume that $r = 1$, and since $W = W_{\langle d\rangle}$ we may fix notation so that $\mathscr{R}(C(d)) = \mathscr{R}^-$. For all i, set $S_i = \langle \mathscr{R}^+(T_i)\rangle$. Then $S_i = O_2(P_i) = S_{i-1}$ for all odd i, by (7.2), so that $C_{S_j}(d) = 1$ for all j. Now $S_1 \cap S_2R = [S_1 \cap S_2R, d] \leqslant [S_2R, d] \leqslant S_2$. Hence $S_1 \cap S_2 = (S_1R \cap S_2R) \cap S_1 = (T_1 \cap T_2) \cap S_1 = O_2(P_2) \cap S_1 \trianglelefteq S_1$. Similarly, $S_1 \cap S_2 \trianglelefteq S_2$. Since $L_2 \nleqslant C(d)$, we have $R \leqslant O_2(P_2)$, and then $\langle S_1, S_2\rangle \in \mathscr{P}$, with $S_1 \cap S_2 = O_2(\langle S_1, S_2\rangle)$.

Set $K_0 = \langle S_1, S_2\rangle$, and put $K_1 = \langle S_3, S_4\rangle$. Then $K_1 = K_0{}^x \in \mathscr{P}$, and since $S_2 = S_3$, we have $(K_0, K_1) \in \varDelta$. By (7.2), $d \in L_0 \cap L_2$, so $d \in K_0 \cap K_1$, and so as $C_{S_i}(d) = 1$ for all i, $\langle K_0, K_1\rangle$ is of type $L_3(2^n)$ for some n, by (4.11). Then $|\mathscr{A}(S_i)| = 2$, so that both members of $\mathscr{A}(S_i)$ are normal in P_i, for i odd. But $S_1 \cap S_2 \in \mathscr{A}(S_1)$, so that $S_1 \cap S_2 \trianglelefteq \langle P_1, K_0\rangle = \langle P_1, P_2\rangle$; whereas $\langle P_1, P_2\rangle \notin \mathscr{P}$.

We conclude that $T_{r,s+1} = 1$. Now suppose that $L_r \neq L_{s+1}$. Since $R \leqslant L_r \cap L_{s+1}$, we then have $W = W_{\langle d\rangle}$, by (6.7). We show that $s - r + 1$ is even in this case. Namely, suppose false, and set $y = x_{s-r}$. Then $L_r{}^y = L_s \leqslant C(d)$. Since also $L_{s+1} \leqslant C(d)$, we have $W = W_d$, by (4.5), and contrary to (7.2). So assume $L_r = L_{s+1}$. Then $x_{2r-1} = x_{2s+1}$, and $1 = x_{2r-1}x_{2s+1} = x_{2(s-r+1)}$. Thus $|W| \mid 2(s - r + 1)$, by (4.1), and so as $|W|$ is divisible by four, $s - r + 1$ is even in this case as well. Hence also $1 = (T_{r,s+1})^{x_{s-r}} = T_{r-1,s}$.

Finally, since $l(T_{r,s}) \leqslant 1 + l(T_{r,s+1})$, by (5.13), we conclude that $T_{r,s} = R$, and this completes the proof of (7.5).

(7.6) LEMMA. *Let $y \in Z(W)\#$, and let $T \in \mathscr{T}$. Then $T \cap T^y = 1$.*

Proof. By (5.1), there exist generators d and d' of D such that $\mathscr{T}_d \neq \mathscr{T}_{d'}$. By (6.6) we may assume that $|W_d| \leqslant 2$. Then $\langle \mathscr{T}_d\rangle$ is not of type $L_3(2^n)$, by (5.3), so $\mathscr{T}_d \leqslant \mathscr{R}$. Let $R \in \mathscr{T}_d$. By (7.5) we have $R = T_{r,s}$ and $T_{r,s+1} =$

$T_{r-1,s} = 1$ for some $r < s$ with $s - r + 1$ even. Then $R \leqslant L_r \cap L_{s+1}$, whence $L_r = L_{s+1}$, by (6.7). Set $x = x_{s-r+1}$. Then $L_r^x = L_r$, so that $[x_{2r-1}, x] = 1$, and so $x \in Z(W)$ as $s - r + 1$ is even. Since $T_r^x = T_{s+1}$ and $R \nleqslant T_{s+1}$, $x \neq 1$, and hence $x = y$. Now as $R \neq R^y$, y is without fixed-points on the orbit R^W of W on $\mathscr{R}$, and $\langle R, R^y\rangle = L_r$.

Take $R^W = \mathscr{R}^+$. Let $T = T_r$, and suppose $T \cap T^y \neq 1$. Set $Q = T \cap T^y$. Since W is transitive on $\mathscr{T}$, by (3.4), we have $Q \in \mathscr{U}_G^*(H\langle y\rangle; 2)$, and since $\langle R, R^y\rangle$ is not a 2-group, we have $\mathscr{R}^+(Q) = \varnothing$, so that in particular, $Q < T$. Set $P = \langle \mathscr{T}(N(Q))\rangle$. Then P admits $H\langle y\rangle$, so that $P \in \mathscr{P}$. Then $P \in \mathscr{P}^*$ and $Q = O_2(P)$, by (3.5). For $S \in \mathscr{T}(P)$ with $N_T(Q) \leqslant S$, we have $l(S) - l(Q) = 1$, by (5.13), and then as $S \cap T$ is an $\mathscr{R}$-string, we have $S = T$. Then $R \leqslant L(P)$, so that $L(P) = L_r$, and $P = TL_r = P_r$. But then $P_{s+1} = P_r^y = P_r$, so that $T_{r-1}^y = T_s = T_r$, and $T_{r,s} = T_r = R$, and so $L_r = P$, and $Q = 1$; a contradiction.

(7.7) LEMMA. *$\langle \mathscr{T}_d\rangle$ is of type $L_3(2^n)$ for some $d \in D\#$.*

Proof. Suppose false. Then $\mathscr{T}_d \leqslant \mathscr{R}$ for all $d \in D\#$ for which $\mathscr{T}_d \neq \{1\}$. Let $T \in \mathscr{R}^+$, $S \in \mathscr{R}^-$. Then $R = T_{r,s}$, $S = T_{r',s'}$, and $T_{r,s+1} = T_{r-1,s} = T_{r',s'+1} = T_{r'-1,s'} = 1$, for some $r < s$, $r' < s'$, by (7.5). Moreover, $s - r + 1$ and $s' - r' + 1$ are even. By replacing S if necessary by a suitable conjugate via $\langle x_2\rangle$, we may assume that $r' - r \leqslant 1$. But $r \neq r'$, since $\mathscr{R}(L_r) \ngeqslant \{R, S\}$. Hence $r' = r + 1$. Suppose $s' < s + 1$. Then $R = T_{r,s} \leqslant T_{r',s'} = S$, which is impossible. Hence $s' \geqslant s + 1$. On the other hand, if $s' > s + 1$ then $s' - 2 \geqslant s$, whence $S^{x_{-1}} = T_{r'-2,s'-2} \leqslant T_{r,s} = R$; again impossible. Hence $s' = s + 1$.

For any $i \in I$, set $\Gamma_i = (T_{r-s+i,i}, T_{r-s+i+1,i+1}, \ldots, T_{i,s-r+i})$. Thus Γ_i is a sequence of conjugates of R and S under W, and successive terms of the sequence belong to distinct orbits. We claim that the terms of each sequence Γ_i are all distinct. Suppose this claim to be false for, say, $i = 1$. Then there exist indices i and j with $1 \leqslant i < j \leqslant s - r + 1$, such that $T_{r-s+i,i} = T_{r-s+j,j}$ so that $T_{r-s+i,j} = T_{r-s+i,i} \cap T_{r-s+j,j} \neq 1$. But even $T_{r-s+i,i+1} = 1$; a contradiction. Thus there are $s - r + 1$ terms in Γ_i, and so $l \geqslant s - r + 1$.

Set $y = x_{s-r+1}$. Let $d \in C_D(R)\#$, $d' \in C_D(S)\#$. Then not both W_d and $W_{d'}$ are of order $2p$, by (6.6), and so by symmetry we may assume $|W_d| \neq 2p$. Then as $R \leqslant L_r \cap L_{s+1}$, we have $L_r = L_{s+1}$, by (6.7). Then $L_r^y = L_r$, whence $[y, x_{2r-1}] = 1$, so that $y \in Z(W)$. Since $T_{r,s+1} = 1 \neq T_{r,s}$, y does not fix T_r, and so as $s - r + 1$ is even, $\{y\} = Z(W)\#$. Hence $x_{2(s-r+1)} = 1$, so that $|W| \mid 2(s - r + 1)$. Thus $|\mathscr{T}| \leqslant 2l$, as W is transitive on $\mathscr{T}$. But now by (6.5) and (7.6), G is not a counter-example to Theorem A.

From now on, we fix $d \in D\#$ with $\langle \mathscr{T}_d\rangle$ of type $L_3(2^n)$. By (7.1), we may fix notation so that $O_2(Z(P_1)) \neq 1$, whence $O_2(Z(P_i)) \neq 1$ for all odd i. Set $U_i = O_2(Z(P_i))$, for all $i \in I$. By (5.8), we may take $U_1 \in \mathscr{X}^+$.

(7.8) LEMMA. *We have* $W = W_{\langle d\rangle}$ *of order* $4p$, *and if* $d' \in D - \langle d\rangle$, *then* $W \neq W_{\langle d'\rangle}$.

Proof. Immediate from (5.3) and (6.6).

(7.9) LEMMA. *Let* $Z \in \mathscr{Z}$ *with* $Z \leqslant C(d)$. *Then* $\langle \mathscr{T}(C(Z))\rangle \notin \mathscr{P}^*$, *so that* $\mathscr{Z}(C(d) = \mathscr{Z}^-$. *Moreover, we have* $l \geqslant 8$.

Proof. Suppose $\langle \mathscr{T}(C(Z))\rangle \in \mathscr{P}^*$. We may then assume that $Z = U_1$ and that $\mathscr{Z}(C(d)) = \mathscr{Z}^+$. Then $U_i \leqslant C(d)$ for all odd i, and then $d \in P_i$ for all odd i, as $\langle \mathscr{T}(C(D))\rangle = 1$, by (3.7). Then also $L_i \leqslant C(D)$ for all even i, by (7.2).

Set $A = U_1U_3$. Then $A \leqslant Z(T_{1,2})$. Since $\langle P_1, P_3\rangle \geqslant \langle P_1, T_2\rangle = \langle P_1, P_2\rangle > P_1$, we have $P_1 \neq P_3$, whence $U_1 \neq U_3$, and $l(A) = 2$. By (3.8) we have $C_T(d) \in \mathscr{T}_d$ for all $T \in \mathscr{T}$, so $C_{P_i}(d) \in \mathscr{P}_d$ for all even i, and $A = O_2(C_{P_2}(d))$, by (5.4). Now, since $P_2 \neq P_4$, we have $C_{P_2}(d) \neq C_{P_4}(d)$, by (3.5), and then since $C_{T_2}(d) = C_{T_3}(d) = C_{O_2(P_3)}(d)$, we have $(C_{P_2}(d), C_{P_4}(d)) \in \Delta$. Hence $O_2(\langle C_{P_2}(d), C_{P_4}(d)\rangle) = 1$, by (4.6), whence $U_1U_3 \neq U_3U_5$. Since also $l(U_3U_5) = l(A^{x_2}) = 2$, we have $U_3 \neq U_5$, and then $U_5 \nleqslant U_1U_3$. Thus $l(AU_5) \geqslant 3$, so that AU_5 is not abelian, by (5.4). Hence $A \nleqslant P_5$. But $A \leqslant P_3$, and $\mathscr{R}^+(P_3) = \mathscr{R}^+(O_2(P_3)) \leqslant \mathscr{R}^+(P_4)$. Thus $A \leqslant P_4$ but $A \nleqslant O_2(P_4)$. Then $U_1 \nleqslant O_2(P_4)$, as $[U_3, U_5] = 1 \neq [A, U_5]$. Hence $U_1 \leqslant L_4$.

We have $T_{1,4} \leqslant C(U_1)$, and $L_4 = \langle U_1, U_1^{x_7}\rangle$. Thus $T_{1,6} = T_{1,4} \cap (T_{1,4})^{x_7}$ centralizes L_4. Then $l(T_{1,6}) \leqslant 1$, by (5.15), whereas $l(T_{1,6}) \geqslant l - 5$, by (5.14). Thus $l \leqslant 6$, and we need only show that $l \geqslant 8$ in order to complete the proof of the Lemma.

We now drop our assumption that there exists $Z \in \mathscr{Z}(C(d))$ with $\langle \mathscr{T}(C(Z))\rangle \in \mathscr{P}^*$, and assume only that $l < 8$. Let $R \in \mathscr{R}$, with $R \nleqslant C(d)$. Then by (7.8) and (7.5) there exist indices r and s with $R = T_{r,s}$, $T_{r,s+1} = T_{r-1,s} = 1$, and $s - r + 1$ even. Then $R \leqslant L_r \cap L_{s+1}$, and so $L_r = L_{s+1}$ by (6.7). Set $R_i = R^{x_{2i}}$ for all i, and set $k = s - r + 1$. As in the proof of (7.7), we have $\{x_k\} = Z(W)\#$, whence $|W| \mid 2k$. Since $T_{r,s} \neq 1$, it follows from (4.7) that $x_j \neq 1$ for any even j, $j < k$, so that in fact $|W| = 2k$. Since $|W| = 4p$ we then have either $k = 6$ or $k \geqslant 10$.

Set $m = \frac{1}{2}k - 1$. We now have $R = R_0 \leqslant T_{r,s}$, $R_1 \leqslant T_{r+2,s}, \ldots, R_m \leqslant T_{r+k-2,s} = T_{s-1,s}$. Supose that $R_0, R_1, \ldots, R_m$ are not all distinct. Then $R_0 = R_i$ for some i, with $1 \leqslant i \leqslant m$, and then $L_r = (L_r)^{x_{2i}} = L_{r+2i}$, where $r < r + 2i < s$. Thus $R \nleqslant O_2(P_{r+2i})$, whereas $R \leqslant T_{r,s} \leqslant O_2(P_{r+2i})$; a contradiction. Hence $R_0, R_1, \ldots, R_m$ are all distinct, and since they are all contained in T_s, where $l(C_{T_s}(d)) = 3$, we have $l \geqslant 3 + k/2$. Since $l < 8$, this yields $k < 10$, whence $k = 6$ and $l \geqslant 6$.

Now, $|W| = 2k = 12$, so that $|\mathscr{T}| \leqslant 12$ and $l \geqslant 1/2\,|\mathscr{T}|$. Then G is not a counter-example to Theorem A, by (6.5) and (7.6).

(7.10) LEMMA. *Let* $Z \in Z^-$. *Then* $l^+(C(Z)) = 4$.

Proof. By (7.9), $d \notin L_1$, so by (7.2) we have $[d, L_1] = 1$ and $d \in L_2$. As $[d, L_1] = 1$, (5.3) implies $C_{P_1}(d) \in \mathscr{P}_d$, and then (3.5) implies that P_1 is the unique member of $\mathscr{P}^*$ containing $C_{P_1}(d)$. As $d \in L_2$, $C_{P_2}(d) \in \mathscr{T}_d$, so there is a bijection of $\mathscr{P}_d$ and P_1^W defined by inclusion. As $U_1 \in \mathscr{Z}^+$ and $P_1 \in \mathscr{P}^*$, $l^+(C(Z))$ is the number of conjugates of P_1 under W containing Z, and hence by our last remark, is also the number of members of $\mathscr{P}_d$ containing Z. Thus it suffices to show that Z is contained in precisely four members of $\mathscr{P}_d$.

Let $T \in \mathscr{T}_d$ with $Z \leqslant Z(T)$. Then $T = \langle \mathscr{T}_d(C(Z)) \rangle$ since $Z(P) = 1$ for any $P \in \mathscr{P}_d$. Since $|\mathscr{P}_d(T)| = 2$, it follows that $\{P \in \mathscr{P}_d : Z \leqslant O_2(P)\}$ is of order two. Suppose $Z \leqslant L = L(P)$, $P \in \mathscr{P}_d$. Then Z is in a unique $S \in \mathscr{T}(P)$ and $C_S(Z) \in \mathscr{A}(S)$. Now $\mathscr{P}_d(S) = \{P, Q\}$ where $Q = \langle \mathscr{T}_d(N(A)) \rangle$. Since $\{T, S\} = \mathscr{T}(Q)$ and $|\mathscr{A}(T)| = 2$ we conclude that $\{P \in \mathscr{P}_d : Z \leqslant L(P)\}$ is of order at most two. By (5.4) (iii) we have equality, which proves (7.10).

(7.11) LEMMA. *We have* $\mathscr{R}^+(P_i) = \mathscr{R}^+(O_2(P_i))$ *for all odd* i.

Proof. For odd i we have $Z(P_i) = U_i \in \mathscr{Z}^+$. Then $d \notin \mathscr{P}_i$ by (7.9), so that $L(P_i) \leqslant C(d)$, by (7.2). Since $\mathscr{R}(C(d)) = \mathscr{R}^-$, we have (7.11).

(7.12) LEMMA. $l < 8$.

Proof. For all odd integers i and j with $i < j$, set $U_{i,j} = U_i U_{i+2} \cdots U_j$. Let k be the largest odd integer for which $U_{1,k}$ is abelian, and set $A = U_{1,k}$. Notice that k exists, as otherwise $U_1 \leqslant P_k$ for all odd k, whence $U_1 \leqslant O_2(P_k)$ for all odd k, by (7.11), and then $U_1 \leqslant \bigcap_I T_i$; contrary to (4.7).

Now AU_{k+2} is not an abelian group, so that $A \nleqslant P_{k+2}$. But $A \leqslant P_k$, and so as $A = \langle \mathscr{Z}^+(A) \rangle$, we have $A \leqslant O_2(P_k) \leqslant T_k \leqslant P_{k+1}$. Since $A^{x_2} = U_3 \cdots U_{k+2}$ is abelian, $U_1 \nleqslant P_{k+2}$, so that $U_1 \nleqslant O_2(P_{k+1})$ and $A \cap O_2(P_{k+1}) = U_3 \cdots U_k$.

Set $P = P_{k+1}$, $Q = O_2(P)$, $V = Z(Q)$, $T = T_k$, $x = x_{2k+1}$, and $K = \langle A, A^x \rangle$. Suppose first that V is not a $\mathscr{Z}$-string. Then $1 \neq C_V(d') \notin \mathscr{Z}$ for some $d' \in D\#$. Suppose $d' \notin \langle d \rangle$. Then $\langle \mathscr{T}_{d'} \rangle$ is not of type $L_3(2^m)$ for any m, by (5.3) and (6.6), and so $\mathscr{T}_{d'} \leqslant \mathscr{R}$. But then H acts irreducibly on $Z(R)$ for any $R \in \mathscr{T}_{d'}$, so that either $C_V(d') = 1$ or $C_V(d') \in \mathscr{Z}$; a contradiction. Thus $d' \in \langle d \rangle$. But $C_T(d) = C_Q(d)$ as $\mathscr{R}^-(P) \leqslant Q$, and so $C_V(d) \leqslant Z(C_T(d))$. By (3.8), $C_T(d) \in \mathscr{T}_d$, and then H acts irreducibly on $Z(C_T(d))$, so $C_V(d) = 1$ or $C_V(d) \in \mathscr{Z}$. Hence V is a $\mathscr{Z}$-string. We note that we have also shown that either $C_V(d) = 1$ or $C_V(d) \in \mathscr{Z}$.

Let $U \in \mathscr{Z}^+(V)$, and let $P^* = \langle \mathscr{T}(C(U)) \rangle$, with $Q \leqslant S \in \mathscr{T}(P^*)$. Then $l(S) - l(Q) = 1$, and $N_S(Q) \leqslant \langle \mathscr{T}(N(Q)) \rangle = P$. But $S \cap P$ is an $\mathscr{R}$-string, by (5.7), so $S \leqslant P$, and so $S = T_k$ or $S = T_{k+1}$. Then $P^* \in \{P_k, P, P_{k+2}\}$, by (6.2). Since $d \in P$, by (7.2), $U \nleqslant Z(P)$, so $P^* = P_k$ or $P^* = P_{k+2}$, and $U = U_k$ or U_{k+2}. Thus $\mathscr{Z}^+(V) = \{U_k, U_{k+2}\}$, and hence $V = ZU_kU_{k+2}$, where $Z = C_V(d)$.

Let $B = A^{x_2} = U_3 \cdots U_{k+2}$. Thus $B \leqslant P_3$ and $B \nleqslant P_1$. It follows as above that $B \leqslant P_2$, that $B \nleqslant O_2(P_2)$, and that $B \cap O_2(P_2) = U_3 \cdots U_k = A \cap B$. Set $V_2 = Z(O_2(P_2))$. Thus $V_2 = \tilde{Z} U_1 U_3$, where $\tilde{Z} = C_{V_2}(d)$. Now $[U_1, U_{k+2}] \leqslant V \cap V_2$, and since $U_1 \nleqslant P_{k+2}$, $[U_1, U_{k+2}] \neq 1$. Since V and V_2 are Z-Strings and Z and $\tilde{Z}$ are trivial or in $\mathscr{Z}$, we then have either $Z = \tilde{Z} \neq 1$, or else $U_3 = U_k$. But if $Z = \tilde{Z}$ then $U_1 \leqslant C(Z)$, whence $\langle \mathscr{Z}^+(T) \rangle \leqslant C(Z)$, and $l^+ \leqslant 4$, by (7.10). Since $l^- = 3$, we get $l \leqslant 7$ in this case, which yields (7.12). Hence we may assume that $U_3 = U_k$. Then $U_1 = U_3^{x_{-2}} = U_{k-2}$, and as $U_1 \nleqslant Q$, we conclude that $k = 3, P = P_4$, and $A = U_1 U_3$. Then $T_{1,4} \leqslant C(A)$, and $T_{1,4} \cap (T_{1,4})^x = T_{1,6} \leqslant C(K)$. Then $l(T_{1,6}) \leqslant 1$, by (5.3) (ii). But $l(T_{1,6}) \geqslant l - 5$, by (5.14), and so $l \leqslant 6$, which yields (7.12).

Now (7.9) and (7.12) provide a contradiction, which completes the proof of Theorem A.

8. Theorem B

Let G be a counter-example to Theorem B. Thus, in addition to satisfying our main Hypothesis, G is of characteristic 2-type, and $N_G(P) \in \mathscr{M}$ for all P in $\mathscr{P}^*$, where $\mathscr{M}$ is the set of maximal 2-local subgroups of G.

Set $L = \langle \mathscr{T} \rangle$, $M = N_G(L)$. Fix T in $\mathscr{T}$, set $Z = Z(T)$, and let $B = (H \cap L)\, T$. Fix $S \in \mathrm{Syl}_2(M)$ with $T \leqslant S$.

(8.1) Lemma. *We have $L = F^*(M)$ isomorphic to $L_3(2^n)$, $S_p(4, 2^n)$, $G_2(2^n)$, ${}^3D_4(2^n)$, ${}^2F_4(2^n)$, $U_4(2^n)$,* or *$U_5(2^n)$ for some n, $n > 1$.*

Proof. Set $X = C_G(L)$. Then $X \leqslant C(P)$ for any $P \in \mathscr{P}^*$, and as $N(P) \in \mathscr{M}$ and G is of characteristic 2-type, we have $X \leqslant O_2(P)$. Hence $X \leqslant O_2(L)$, and so $X = 1$, by our Hypothesis. In particular, $Z(L) = 1$. By Theorem A, L is a split BN-pair of rank two, with Cartan subgroup $H \cap L$. It then follows from Theorem A of [3] that L is isomorphic to one of the listed groups (with n arbitrary). Since the field of definition of $P/O_2(P)$ is larger than $\mathbb{Z}_2$ for all $P \in \mathscr{P}$, we have $n > 1$.

(8.2) Lemma. *Either $D \leqslant L$ or $p = 3$ and $LD \simeq PGL(3, 4^n)$.*

Proof. Since $D \leqslant C(H \cap L)$ where $H \cap L$ is a Cartan subgroup of L, D induces inner-diagonal automorphisms on L. Since $C(L) = 1$, we then either have (8.2) or $p = 5$ and $LD \simeq PGU(5, 4^n)$, n odd. But $p \mid 4^n - 1$, by (3.1), so this last case is excluded.

(8.3) Lemma. *We have $N(T) \leqslant M$ and $N(P) \leqslant M$ for any $P \in \mathscr{P}(T)$.*

Proof. Let $P \in \mathscr{P}(T)$, and put $N = N(P)$. Since $T \in \mathrm{Syl}_2(P)$, by (3.1), we have $N = PN_N(T)$, by the Frattini argument, and hence it will suffice to show that $N(T) \leqslant M$. By (6.2), $|\mathscr{P}(T)| = 2$, and then also $L = \langle \mathscr{P}(T) \rangle$, by (3.2) and (6.3). Thus, we need only show that $N(T)$ fixes $\mathscr{P}(T)$. Assume false. Since $\{N(P) \colon P \in \mathscr{P}(T)\} \leqslant \mathscr{M}$, it follows that no non-idendity characteristic subgroup of T is normal in P, for some P in $\mathscr{P}(T)$. Then by (6.1), and inspection, L is isomorohic to $L_3(2^n)$ or $S_P(4, 2^n)$. In both of these cases we have $|\mathscr{A}(T)| = 2$, and $\mathscr{A}(T) = \{O_2(P) \colon P \in \mathscr{P}(T)\}$. Thus $N(T)$ fixes $\mathscr{P}(T)$ as a set, which yields (8.3).

(8.4) LEMMA. *Let $P \in \mathscr{P}(T)$, $Q = O_2(P)$, and suppose $Q \notin \mathit{И}^*_M(P; 2)$. Then $L \simeq U_4(2^n)$, $P = O^{2'}(C_L(Z))$ and for any $R \in \mathit{И}^*_M(P; 2)$, we have*

(i) *$Z = Z(R)$, and*

(ii) *R is not invariant under $C_L(Z)$.*

Proof. Let $R \in \mathit{И}^*_M(P; 2)$, and let $x \in R - Q$. Then $x \in M - L$, so that x induces a non-inner automorphism on L, by (8.1). As L is simple, we may regard $L\langle x \rangle$ as a subgroup of $\mathrm{Aut}(L)$ in the natural way. By [12] (Theorems 30 and 36) we have $x = idgf$, where $i \in L$, and where d, g, and f are, respectively, diagonal, graph, and field automorphisms of L which leave T and $H \cap L$ fixed. Since $P^x = P$, we have $g = 1$, and since x is a 2-element, so is f. Since x normalizes T, so does i, whence $i \in B = (H \cap L)\, T$. We have df conjugate to f in $N_{\mathrm{Aut}(L)}(T)$, so we may assume without loss that $d = 1$. Thus $x = if$.

Set $Y = N_{\mathrm{Aut}(L)}(P)$, and put $Y = Y/Q$. Then $[\bar{P}, \bar{x}] = 1$, so that $\bar{f} = \bar{\imath}^{-1}$. Since $\bar{P}$ is a Bender group, and $\bar{f}$ induces a field automorphism on $\bar{P}$, it follows that $[\bar{P}, \bar{f}] = 1$, whence L is a twisted group. Then as $|\bar{f}|$ is a power of two, L is a unitary group. If $L \simeq U_5(2^n)$ then $\bar{P} \simeq L_2(2^{2n})$ or $SU_3(2^n)$, so that $[\bar{P}, \bar{f}] \neq 1$. Thus $L \simeq U_4(2^n)$ and $\bar{P} \simeq L_2(2^n)$, which identifies P as $O^{2'}(C_L(Z))$, by inspection of the maximal parabolic subgroups of $U_4(2^n)$. Again by inspection, $Z = Z(Q)$.

We have $|Z| = 2^n$, and Z is a root-subgroup of T, whence Z admits and then centralizes f. Suppose $x \notin C(Z)$. Then $i \notin C(Z)$. But $\bar{\imath} \in C_{\bar{B}}(\bar{P}) = C_{\overline{H \cap L}}(\bar{P})$, a cyclic group of order $2^{2n} - 1$ which acts transitively on $Z\#$. Since x is a 2-element, and $|\bar{\imath}|$ is odd, $\bar{f}$ inverts $\bar{\imath}$, $|\bar{\imath}| \mid 2^n + 1$, and i centralizes Z; a contradiction. Thus (i) holds. Again as $\bar{f}$ inverts $C_{\overline{H \cap L}}(\bar{P}) \cap C(Z)$, (ii) holds.

(8.5) LEMMA. *Let $P \in \mathscr{P}(T)$, and let $R \in \mathit{И}^*_G(P; 2)$. Then $R \leqslant M$.*

Proof. Set $R_0 = N_R(O_2(P))$. Then $R_0 \leqslant N(P)$, as $N(P) \in \mathscr{M}$, and then $R_0 \leqslant M$, by (8.2). Then $Z(R_0) = Z(O_2(P))$, by (8.4). Now $N(R_0) \leqslant N(P)$, again as $N(P) \in \mathscr{M}$, whence $R_0 = R \leqslant M$.

(8.6) LEMMA. *$L \not\simeq L_3(2^n)$ and $L \not\simeq S_P(4, 2^n)$.*

Proof. Suppose false, and let $P \in \mathscr{P}(T)$. Let $P \leqslant N \in \mathscr{M}$. Then $O_2(N) \leqslant O_2(P)$, by (8.4) and (8.5). But P acts irreducibly on $O_2(P)/Z(P)$, so that, as G

is of characteristic 2-type, we have $O_2(N) = O_2(P)$. Since $N(P) \in \mathscr{M}$, this yields $P \trianglelefteq N$. Thus $N(P)$ is the unique member of $\mathscr{M}$ which contains P.

It follows from [1] (Theorem 3) that either $L = F^*(G)$ or that G has sectional 2-rank four. Since Theorem B is assumed false, we then have $LD \cong PGL(3, 4)$ and the sectional 2-rank of G is four. Moreover, $F^*(G) \ncong L_3(4)$, so $T \notin \mathrm{Syl}_2(G)$, by the results of Gorenstein and Harada in [7]. Since G is of characteristic 2-type, it then follows from Theorems A and D of part II of [8] that $F^*(G) \cong M_{23}$ or HJM, where HJM is the Hall–Janko–McKay group of order $2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$. Since M_{23} has no outer automorphisms of order 3, and M_{23} has no subgroup isomorphic to $PGL(3, 4)$, we then have $F^*(G) \cong HJM$. Then by [4] (lemma 3.5 and the proof of lemma 3.6) we have $m_3(C_G(D)) = 3$. Since $N_G(L)$ contains no elementary abelian subgroups of order 27, this contradicts $H \leqslant N(L)$. Thus $F^*(G) \ncong HJM$, and the lemma is proved.

(8.7) Lemma. *Let* $L \cong G_2(2^n)$, $P = \langle \mathscr{T}(C(Z)) \rangle$, $Q = O_2(P)$, $\bar{P} = P/Q$, *and* $\tilde{Q} = Q/Z$. *Then*

(i) $\bar{P} \cong L_2(2^n)$

(ii) $\Phi(\tilde{Q}) = 1$, *and* $m(\tilde{Q}) = 4n$.

Proof. One may verify (i) and (ii) directly from the Steinberg relations for $G_2(2^n)$.

(8.8) Lemma. *Let* $L \cong {}^3D_4(2^n)$, $P = \langle \mathscr{T}(C(Z)) \rangle$, $Q = O_2(P)$, $\bar{P} = P/Q$, $\tilde{Q} = Q/Z$. *Then*

(i) $\bar{P} \cong L_2(2^{3n})$,

(ii) $\Phi(\tilde{Q}) = 8n$, *and*

(iii) $\tilde{Q}$ *is an irreducible module for* $\bar{P}$.

Proof. Parts (i) and (ii) follow as in (8.7), and also $[\tilde{Q}, \bar{P}] = \tilde{Q}$ and $C_{\tilde{Q}}(P) = 1$. As before, $\tilde{Q}$ is a module for $\bar{P}$ over $GF(2^n)$. If $\tilde{Q}$ has a proper irreducible $\bar{P}$-submodule, then $m(\tilde{Q}) \geqslant 12n$, which is not the case, and so $\tilde{Q}$ is irreducible.

(8.9) Lemma. *Let* $L \cong {}^2F_4(2^n)$, $P = \langle \mathscr{T}(C(Z)) \rangle$, $Q = O_2(P)$, *and* $\bar{P} = P/Q$. *Then*

(i) $\bar{P} \cong Sz(2^n)$

(ii) $|\bar{Q}| = 2^{10n}$,

(iii) P *has two non-central* 2-*chief factors, each of rank* $4n$, *and*

(iv) P *has a unique maximal abelian normal subgroup* E, *and*

(a) $\Phi(E) = 1$, $m(E) = 5n$,

(b) $[Q, E] = Z$, *and* $\bar{P}$ *acts irreducibly on* E/Z.

Proof. All of the above, with the exception of the uniqueness of E, follow without difficulty from the analogues of the Chevelley commutator relations obtained by Ree for the groups ${}^2F_4(2^n)$; and we refer the reader to [10] (section 2) for verification. The group E is just $[Q, Q]$. Moreover $|P: O^2(P)| = 2^n$, and $C_Q(O^2(P))$ is a Suzuki 2-group of order 2^{2n}. Thus, if E is not unique, then there exists $F \leqslant P$, with F abelian, $[F, O^2(P)] \neq 1$, and $E \cap F = Z$. Then also $|F| \geqslant 2^{5n}$, and $Q = EFC_Q(O^2(P))$. But then $cl(Q) = 2$, which is not the case. Thus (8.9) holds.

(8.10) Lemma. *Let $z \in Z\#$. Then $C_G(z) \leqslant M$.*

Proof. Suppose false, and put $N = C_G(z)$, $R = O_2(N)$, and $X = C_L(z)$. Thus $P = \langle \mathscr{T}(C(Z)) \rangle = O^{2'}(X)$. By (8.5), $R \leqslant M$, and by (8.4) (ii), $R \leqslant O_2(P)$. Since G is of characteristic 2-type, we have $C_Q(R) \leqslant R$, for $Q = O_2(P)$. Since $N(P) \in \mathscr{M}$ and $N(P) \leqslant M$, by (8.3), no non-identity characteristic subgroup of R is also characteristic in Q. In particular, $R < Q$.

Now Q/Z is not an irreducible module for P/Q. For $L = U_4(q)$ or $U_5(q)$, however, we have $m(Q/Z) = 2n$ or $6n$, and $P/Q \cong L_2(2^n)$ or $SU(3, 2^n)$, respectively, so $L \ncong U_4(2^n)$ and $L \ncong U_5(2^n)$. By (8.8) (iii), $L \ncong {}^3D_4(2^n)$. Suppose $L \cong G_2(2^n)$. Since $Z < R < Q$, it then follows that $m(R/Z) = 2^{2n}$, and that R/Z is irreducible for P/Q. If $Z = Z(R)$, then as $Z = Z(Q)$ we have Z char R and Z char Q; a contradiction. Hence $Z < Z(R)$, and so R is abelian. Then $\Phi(R) = 1$, whereas P has no normal abelian subgroups of rank $3n$, by [14] (lemma 3.11).

Hence $L \cong {}^2F_4(2^n)$. We adopt the notation of (8.9). Suppose first that $R \geqslant E$. Then $Z(R) \leqslant E$ as E is a maximal normal abelian subgroup of P. Then as $Z < Z(R)$, and as P/Q acts irreducibly on E/Z, we have $E = Z(R)$. But then $N(Z(R)) \geqslant \langle N(P), N(R) \rangle$, and as $N(P) \in \mathscr{M}$, $P \trianglelefteq N(R)$, and then $N(R) \leqslant M$; a contradiction. Thus $E \nleqslant R$, and so $E \cap R = Z$. Since $|Q| = 2^{10n}$, we have $|R| \leqslant 2^{6n}$, and P has just one non-central chief factor in R.

Set $S = [R, P]\, Z$, and $U = Z(R)$. Then $|S| = 2^{5n}$, and as $S \neq E$, and E is unique, S is non-abelian. Hence $U \cap S = Z$ and $P = C_P(U)\, Q$. But then U centralizes $[E, C_P(U)]\, Z = E$, so that $U \leqslant E$ and $R \cap E > Z$; a contradiction.

(8.11) Corollary. $S \in \mathrm{Syl}_2(G)$.

Proof. Let $S \leqslant S^* \in \mathrm{Syl}_2(G)$, and let $z \in Z(S^*)$. Then $z \in N(Z) \leqslant N(P)$, where $P = \langle \mathscr{T}(C(Z)) \rangle$. We have $Z = Z(O_2(P))$, and then as G is of characteristic 2-type, $z \in Z$. Hence $S^* \leqslant M$, by (8.10), whence $S = S^*$.

(8.12) Lemma. *Suppose $L \cong U_4(2^n)$. Then $S = T$.*

Proof. Suppose false, and let $x_1 \in S - T$ with $x_1{}^2 \in L$. Regarding $L\langle x_1 \rangle$ as a subgroup of $\mathrm{Aut}(L)$, we have $x_1 = if$ for some $i \in L$, and for f a field automorphism of L of order two, where f fixes T and $H \cap L$. Thus $L\langle x_1\rangle = L\langle x\rangle$, where $|x| = 2$ and x induces a standard field automorphism of L, of order two, and where we may assume $x \in S$.

Now $C_L(x) \cong S_P(4, 2^n)$, and then as G is of characteristic 2-type, $|O_2(C_G(x))| > 2^{4n}$. Hence $|S| > 2^{8n}$. But $|T| = 2^{6n}$, so that $|S| \leqslant n \cdot 2^{6n}$; a contradiction.

From now on, we fix $P = \langle \mathscr{T}(C(Z))\rangle$.

(8.13) LEMMA. *L has two conjugacy classes of involutions. If z and x are representatives of these classes, with $z \in Z$, and if we put $Y = O^{2'}(C_L(x)$, then $Y/O_2(Y) \cong L_2(2^n)$, and the following table describes $O_2(Y)$, and compares $|O_2(Y)|$ with $|O_2(P)|$, when $L \ncong {}^2F_4(2^n)$.*

L	$\lvert O_2(Y)\rvert$	$\lvert Z(O_2(Y))\rvert$	$\lvert O_2(P)\rvert$
$U_4(2^n)$	2^{4n}	2^{4n}	2^{5n}
$U_5(2^n)$	2^{8n}	2^{4n}	2^{7n}
$G_2(2^n)$	2^{3n}	2^{3n}	2^{5n}
${}^3D_4(2^n)$	2^{9n}	2^{5n}	2^{9n}

Proof. For $L \cong U_5(2^n)$, $G_2(2^n)$, or ${}^3D_4(2^n)$, se [16], [15], and [17], sections 7, 8, and 8, respectively. For $L \cong U_4(2^n)$, use direct calculation.

(8.14) LEMMA. *Let $z \in Z\#$, $g \in G$, with $x = z^g \in L$. Then $g \in M$.*

Proof. Suppose false. Then $z \notin x^L$, by (8.10). Set $Y = O^{2'}(C_L(x))$. Then $Y \leqslant C_{M^g}(x)$, by (8.10), so that $Y \leqslant N_{M^g}(P^g)$. Since $L^g = F^*(M^g)$, and since $\mathrm{Out}(L)$ is solvable, it follows from (8.13) that $O^2(Y) \leqslant P^g$. Then $L \ncong {}^2F_4(2^n)$, as $Y/O_2(Y)$ is not a 3'-group.

Now $Y/O_2(Y) \cong L_2(2^n)$, so that either $P^g \subseteq O_2(P)\, Y$ or else $L \cong {}^3D_4(2^n)$ with $P/O_2(P) \cong L_2(2^{3n})$, or $L \cong U_5(2^n)$ with $P/O_2(P) \cong U_3(2^n)$. Hence either $O_2(Y) \subseteq C(P^g/O_2(P^g))$ or $L \cong U_5(2^n)$ and $|O_2(Y): C_{O_2}(P^g/O_2(P^g))| = 2$. By (8.4) and (8.12) we then have $|Y: Y \cap P^g| \leqslant 2$, and equality holds only if $L \cong U_5(2^n)$. But table (8.13) shows that $|O_2(Y)|/2 > |O_2(P)|$ if $L \cong U_5(2^n)$, so $L \ncong U_5(2^n)$, and $Y \subseteq P^g$.

Suppose $L \cong U_4(2^n)$. Then $O^{2'}(N_L(O_2(Y))) \in \mathscr{P}^*$. Also $O_2(P^g)$ normalizes $O_2(Y)$ as $Z(P^g) = Z(Y)$ where $cl(O_2(P^g)) = 2$. Since $N_G(Q)$ is a maximal 2-local subgroup of G for any $Q \in \mathscr{P}^*$, we then have $N(O_2(Y)) \subseteq M$, by (8.3), and so $P^g = O_2(P^g)\, Y \subseteq M$. Then $P^g \subseteq L$, by (8.12), and $P^g \in \mathscr{P}(T)$ with $P^g \neq P$. But then $P^g \ncong P$; a contradiction. Thus $L \ncong U_4(2^n)$.

Now if $L \cong {}^3D_4(2^n)$ then $|O_2(Y)| = |O_2(P)|$ so that $O_2(Y) = O_2(P^g)$. But in this case $Z(O_2(Y)) > Z \neq Z^g$; a contradiction.

The one remaining case is that in which $L \simeq G_2(2^n)$. Here $|O_2(Y)| = 2^{3n}$, by (8.13), and $O_2(Y)$ is abelian. Then $\Phi(O_2(Y)) = 1$, whereas P has no normal abelian subgroups of rank $3n$, as was shown in the proof of (8.10). Thus we have a contradiction in this case as well, which proves (8.14).

(8.15) LEMMA. *Let $z \in Z\#$, $g \in G$, with $x = z^g \in M$. Then $g \in M$.*

Proof. Without loss, we may assume that $x \in S$, so that Z admits x. Suppose first that $|C_Z(x)| \geqslant 4$. Since $C(x) \leqslant M^g$ and since $\mathrm{Out}(L^g)$ has cyclic Sylow 2-subgroups, we then have $Z \cap L^g \neq 1$. But then $g \in M$, by (8.14). Hence $n = 2$, and $S = \langle T, x\rangle = \langle T, f\rangle$ where f induces a field automorphism on L and where $|f| = 2$. If $L \simeq U_4(4)$ or $U_5(4)$ we have $[Z, f] = 1$, so $L \simeq G_2(4)$ or ${}^3D_4(4)$. Let $\{P_1\} = \mathscr{P}(T) - \{P\}$, and set $V = Z(O_2(P_1))$. Then V is a natural module for $P_1/O_2(P_1)$, with $P_1/O_2(P_1) \simeq L_2(2^n)$. Thus $z^L \geqslant V\#$. But x fixes P_1 as $P_1 \not\simeq P$, and so $|C_V(x)| \geqslant 4$, $C_V(x) \cap L^g = 1$, and (8.14) yields a contradiction.

Now by (8.15) and (8.10), z is contained in a unique conjugate of M, for every involution $z \in M$ such that z is central in Sylow 2-subgroup of G. By a Theorem of Holt [9], it follows that $M \trianglelefteq G$, and then $L = F^*(G)$ by (8.1). This completes the proof of Theorem B.

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